Conjectural Normal Form for Elements of Coxeter Groups

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Preliminaries and Terminology

(W, S) is a Coxeter System with $S = \{s_1, \ldots, s_n\}$.

$m(s_i, s_j) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ $\qquad$ $(s_i s_j)^{m(s_i, s_j)} = 1$

$T := \bigcup_{u \in W} uSu^{-1}$ (reflections)

$\ell_{(W,S)} : W \to \mathbb{N}$ (length function)

$N(u) = \{t \in T \mid \ell(tu) < \ell(u)\}$ (reflection cocycle)

$h_{(W,S)} : T \to \mathbb{N}$ (height function)

$$h_{(W,S)}(t) = \frac{\ell_{(W,S)}(t) - 1}{2}$$

$W' \leq W$ is a *reflection subgroup* if $W' = \langle W' \cap T \rangle$.

Fact (Dyer and Deodhar): $(W', \chi(W'))$ is a Coxeter system where

$$\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}$$

Preliminaries and Terminology

V, Φ, Φ^+, Π - Standard Reflection Representation (root system)

$B : V \times V \to \mathbb{R}$ given by $B(\alpha, \beta) = -\cos\frac{\pi}{m(s_\alpha, s_\beta)}$ (or ≤ -1)

$W' \leq W$, we get $\Phi_{W'}$, $\Phi^+_{W'}$, $\Pi_{W'}$ all subsets of Φ.

For any $u \in W$, $\Phi_u := \Phi^+ \cap u(-\Phi^+)$

Bijections:

$\Phi^+ \leftrightarrow T$

$\Pi \leftrightarrow S$.

$\Phi_w \leftrightarrow N(w)$.

Dihedral Reflection Subgroups and Height

A reflection subgroup, W', is called *dihedral* if $|\chi(W')| = 2$.

$\mathscr{M} := \{W' \leq W |\ W'$ is a maximal dihedral reflection subgroup$\}$

For $t \in T$, $\mathscr{M}_t := \{W' \in \mathscr{M} \mid t \in W'\}$.

Lemma

For any $t \in T$, we have $h_{(W,S)}(t) = \sum_{W' \in \mathscr{M}_t} h_{(W',\chi(W'))}(t)$

Definition

For any $t \in T$, we have

$$h^{\infty}(t) := \sum_{\substack{W' \in \mathscr{M}_t \\ |W'| = \infty}} h_{(W',\chi(W'))}(t)$$

Dominance Order and Infinite Height

We say that $\alpha \in \Phi$ *dominates* $\beta \in \Phi$ written $\beta \preceq \alpha$ if, for all $w \in W$, $w(\alpha) \in \Phi^-$ implies that $w(\beta) \in \Phi^-$.

- $\preceq$ is a partial order.
- Can restrict to Φ^+ and transfer to T.

Theorem

For any $t \in T$, $h^\infty(t)$ represents the number of reflections strictly dominated by t.

Remark

Brink and Howlett used minimal elements (i.e. $h^\infty(t) = 0$) to construct automata to prove that (W, S) is automatic.

Partitioning the Reflections

Define $T_n := \{t \in T \mid h^\infty(t) = n\}$

Under the bijection $T \leftrightarrow \Phi^+$, we get the corresponding set Φ_n^+.

Theorem (Dyer and Fu)

For all $n \in \mathbb{N}$, T_n (and thus Φ_n^+) is finite.

Interesting Sets:

$T_{\leq m} := \bigcup_{n \leq m} T_n$

$\Phi_{\leq m}^+$ (under bijection)

$N_m(w) = N(w) \cap T_{\leq m}$

2-Closure in the Root System

Definition

Let $\Gamma \subseteq \Phi^+$. We say that Γ is *2-closed* if for all $\alpha, \beta \in \Gamma$, we have $(\mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}\beta) \cap \Phi^+ \subseteq \Gamma$.

1. While this can be stated in terms of the reflections only, we say $T' \subseteq T$ is 2-closed if the corresponding subset of Φ^+ is 2-closed.
2. There are other types of closure.

Definition

We say a subset $\Gamma \subseteq \Phi^+$ is *biclosed* if Γ and $\Phi^+ \setminus \Gamma$ are both 2-closed.

For any subset $\Psi \subseteq \Phi^+$, let $\overline{\Psi}$ be the 2-closure.

Subsets of Roots

Example

For all $u \in W$, Φ_u (and thus $N(u)$) is biclosed.

Definition

Let $\Gamma \subseteq \Phi^+$.

1. We say Γ is *balanced* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\beta \in \Phi^+_{W'}$ such that $l_{(W',\chi(W'))}(s_\beta) < l_{(W',\chi(W'))}(s_\alpha)$ implies that $\beta \in \Gamma$.
2. We say Γ is *bipedal* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$ with $\alpha \notin \Pi_{W'}$, $\Pi_{W'} \subset \Gamma$.
3. We say Γ is *unipedal* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\Pi_{W'} = \{\beta, \gamma\}$ implies that either $\beta \in \Gamma$ or $\gamma \in \Gamma$.

Closure Conjectures

The following conjectures and results are due to Dyer.

Conjecture

If $\Gamma \subseteq \Phi^+$ is unipedal, then $\overline{\Gamma}$ is biclosed.

Conjecture

Let $A := \{\Gamma \subseteq \Phi^+ | \ \Gamma \text{ is biclosed}\}$. Then A is a complete lattice with $\bigvee_{i\in I} \Gamma_i = \overline{\bigcup_{i\in I} \Gamma_i}$ for an arbitrary family $\{\Gamma_i\}_{i\in I} \subset A$.

Conjecture

Let $m \in \mathbb{N}$ and let $\Phi^+_{\leq m} = \{\alpha \in \Phi^+ | \ h^\infty(s_\alpha) \leq m\}$ be the subset of positive roots corresponding to $T_{\leq m}$. Then $\Phi^+_{\leq m}$ is balanced (and hence bipedal).

Normal Form

Theorem

Let $m \in \mathbb{N}$ such that $\Phi^+_{\leq m}$ is bipedal.

1. *For any $x \in W$, there is a unique $x' \in W$ such that $N_m(x') = N_m(x)$, and any $y \in W$ with $N_m(y) \supseteq N_m(x)$ can be written $y = x'y''$ with $l(y) = l(x') + l(y'')$.*
2. *Any $1 \neq w \in W$ can be uniquely written as $w = w_1 \cdots w_n$ for certain $w_i \in W \setminus \{1\}$ with $i = 1, ..., n$ and $n \geq 1$ such that $l(w) = l(w_1) + \cdots + l(w_n)$ and $w_i = (w_i \cdots w_n)'$ for each i where $x \mapsto x'$ is as in 1.*

Proposition

Let (W, S) be a Coxeter system, and let $m \in \mathbb{N}$.

1. *If (W, S) is finite or affine, then $\Phi^+_{\leq m}$ is bipedal.*
2. *If (W, S) is right-angled, then $\Phi^+_{\leq m}$ is bipedal.*