DOMINANCE AND REGULARITY IN COXETER GROUPS

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Abstract

by

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The main purpose of this thesis is to generalize many of the results of Dyer, [18]. Dyer introduces a large family of finite state automata that are demonstrated to recognize many natural subsets of Coxeter groups. In the current work, we consider generalized length functions in Coxeter systems and show that these length functions lead to a larger family of finite state automata, which in turn recognize a larger class of natural subsets of Coxeter groups. By a well-known result, these subsets (called regular subsets) will then have a rational multivariate Poincaré series.

Dominance order on the set of reflections (or positive roots) of a Coxeter system plays a major role in the study of regular subsets. We draw the connection between dominance in root systems and a notion of closure in root systems. We describe some conjectures, due to Dyer, [17], and introduce a conjectural normal form for Coxeter group elements related to closure and dominance.

In [14], Dyer introduced the notion of the imaginary cone to characterize dominance. We investigate the imaginary cone in the special case of hyperbolic Coxeter systems. We show that any infinite, irreducible, non-affine Coxeter system has universal reflection subgroups of arbitrarily large rank. This allows us to deduce some consequences about the growth type of Coxeter systems.

For Laura, Sue, Kristin, and Mark

CONTENTS

FIGURES

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SYMBOLS

W	Coxeter group
S	Coxeter group generators
l	Standard length function on W
l_p	General length function on W
T	Reflections
N	Reflection cocycle
Φ	Root system
Φ^+	Positive roots
Π	Simple roots
$(\, , \,)$	Bilinear pairing on root system
$\leq_A$	Twisted Bruhat order on group
W'	Reflection subgroup
$\mathcal{M}$	Monomials in $\mathbb{Z}[\mathbf{X}]$
$\preceq$	Dominance order on reflections
$\leq$	Weak order on T; Order on $\mathcal{M}$
ℓ_p	General length function on S^*

CHAPTER 1

INTRODUCTION

In this dissertation, we investigate some of the applications of dominance order on the root system (or reflections) of a Coxeter group. Dominance was first defined by Brink and Howlett in [5], where it is used to show that Coxeter groups are automatic. Brink further investigates the dominance minimal roots in [4].

Despite this work, the dominance order is still poorly understood in general. More recently, in [15] Dyer has employed the idea of using the imaginary cone as a combinatorial tool for studying dominance in the root system. Our general goal will be to further the understanding of the role dominance plays in different aspects of study in Coxeter systems. The dissertation is organized as follows.

Chapter 2 introduces the basic background information necessary to develop the results in the later chapters. These ideas will be used throughout the thesis. The thesis is broken up into three independent parts, which are loosely related through their connections to dominance order.

In Chapter 3, we examine general length functions on Coxeter systems. These give rise to a class of height functions, which we demonstrate can be understood by studying the weak order and give a nice characterization of dominance order. These height functions allow us to define certain finite sets of reflections that partition the entire set of reflections. We begin Chapter 4 by defining regular languages and finite state automata. We are then able to use the partition of the reflections described in Chapter 3 to define a class of canonical finite automata for the Coxeter system. These automata generalize the automata defined by Dyer in [18]. In Chapter 5, we use the automata defined in Chapter 4 to describe large classes of regular subsets of Coxeter groups, i.e. sets that can be recognized by a finite state automaton. We introduce different types of regularity; in particular, our notion of p-complete regularity allows us to easily describe certain regular subsets of a Coxeter group. Regularity has some interesting and important implications for (multivariate) Poincaré series of subsets of a Coxeter system, and we will discuss these briefly. Due to the fact that many natural subsets of Coxeter systems cannot be described as completely regular in the sense of Chapter 5, we use Chapter 6 to define an alternate type of regularity, which will allow us to demonstrate the regularity of many subsets of the set of reflections in a Coxeter system. We end with a characterization of regular sets inside of products $W^n \times T^m$ where W is a Coxeter system with T the set of reflections. Finally, in Chapter 7, we describe how the finite sets that give regularity also lead to a large family of modules for the generic Iwahori-Hecke algebra.

Next, in Chapters 8 and 9, we discuss a notion of closure in the root system (or reflections). Chapter 8 classifies possible partial orders on a Coxeter system, generalizing Bruhat order, using certain closed subsets of the roots. These results appear in [20] as well. Chapter 9 focuses on discussing closure in general and other types of subsets of roots. We outline some conjectures, posed by Dyer (see [17]), involving closure in the root system. We also describe an application of these conjectures related to closure of the sets defined in Chapter 4, which would lead to a normal form for elements of the Coxeter group.

Finally, in Chapters 10-12, we investigate some properties of the imaginary cone, especially in hyperbolic Coxeter systems. In Chapter 10, we provide some characterizations of hyperbolic Coxeter systems. As a byproduct of our characterizations, we obtain a simple, case-free proof of the well known fact that any Coxeter system contains a hyperbolic subsystem as a standard parabolic subsystem. Chapter 11 focuses on studying the imaginary cone in the case of hyperbolic Coxeter systems. We use the imaginary cone to show that any Coxeter system contains universal reflection subgroups of arbitrarily large rank. Furthermore, in the hyperbolic case, the positive spans of the simple roots of the universal reflection subgroups are shown to approximate the imaginary cone (using an appropriate topology on the set of roots). This answers in the affirmative a question due to Dyer [15] in the special case of hyperbolic Coxeter systems. Lastly, Chapter 12 uses the results from Chapter 11 describing large universal reflection subgroups in any Coxeter system to extend the results of [29] regarding exponential growth in parabolic quotients in Coxeter groups.

CHAPTER 2

BACKGROUND MATERIAL

We begin with a chapter introducing the basic objects and terminology used throughout the thesis.

2.1 Coxeter Groups

Definition 2.1.1. We define a *Coxeter system* to be a pair (W, S) consisting of a group W and a set of generators $S \subset W$ subject only to relations of the form $(ss')^{m(s,s')} = 1$ where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$. If s and s' have no such relation, we say that $m(s,s') = \infty$.

We may sometimes abuse terminology and call (W, S), or W, a Coxeter group.

For general results about Coxeter groups, we refer the reader to [23] or [2]. In general, S does not need to be finite. For many of the results we will state we need S to be finite. We shall specify when this is the case.

Every element $w \in W$ can be written as a product of the generators $w = s_1 s_2 \cdots s_m$, each $s_i \in S$. If m is minimal among all such expressions for w we will say that $s_1 s_2 \cdots s_n$ is a *reduced expression* for w and we say that w has length n, i.e. $l(w) = n$. So $l : W \to \mathbb{N}$ denotes the standard length function on (W, S).

We let T denote the set of W-conjugates of S, $T := \bigcup_{w \in W} wSw^{-1}$, and call T the set of reflections. Then, following [19] we define $N : W \to \mathscr{P}(T)$ to be the "reflection cocycle" given by $N(w) = \{t \in T | \; l(tw) < l(w)\}$. We call $N(w)$ a cocycle because it satisfies the cocycle condition:

$$N(xy) = N(x) + xN(y)x^{-1}$$

where $+$ represents symmetric difference on the power set of T, $A + B = (A \cup B) \setminus (A \cap B)$. The existence of this cocycle implies that W acts on $\mathscr{P}(T)$ by $w \cdot A = N(w) + wAw^{-1}$ for $A \subseteq T$.

2.2 Root Systems

Without loss of generality, we will assume that (W, S) is realized in its standard reflection representation on a real vector space V with Π denoting the set of simple roots and $(-,-) : V \times V \to \mathbb{R}$ representing the associated bilinear form given by $(\alpha, \beta) = -\cos\frac{\pi}{m(s_\alpha, s_\beta)}$ if $m(s_\alpha, s_\beta) \neq \infty$ and $(\alpha, \beta) \leq -1$ otherwise. Then we denote the roots and positive roots by Φ and Φ^+ respectively. Recall that $\Phi = \Phi^+ \dot{\cup} - \Phi^+$, and we may sometimes write $\Phi^- := -\Phi^+$.

For $\alpha \in \Phi$ we let $s_\alpha \in T$ represent the corresponding reflection. Recall that the set of reflections are in bijective correspondence with the set of positive roots under the bijection $\alpha \mapsto s_\alpha : \Phi^+ \to T$. For $t \in T$, we will let $\alpha_t \in \Phi^+$ denote the unique positive root α with $s_\alpha = t$. We recall the following two facts

$$l(wt) < l(w) \Leftrightarrow w(\alpha_t) \in \Phi^- \text{ for } t \in T \text{ and } w \in W \tag{2.2.1}$$

$$l(tw) < l(w) \Leftrightarrow w^{-1}(\alpha_t) \in \Phi^- \text{ for } t \in T \text{ and } w \in W \tag{2.2.2}$$

We can then describe the W-action on Φ in terms of the length function l by the following formula

$$w(\pm\alpha_t) = \pm\epsilon\alpha_{wtw^{-1}},\ \epsilon = \begin{cases} 1, & \text{if } l(wt) < l(w) \\ -1, & \text{if } l(wt) > l(w) \end{cases} \tag{2.2.3}$$

for any $w \in W$ and $t \in T$.

2.3 Reflection Subgroups

We call a subgroup W' of W a *reflection subgroup* of W if it is generated by the reflections it contains, $W' = \langle W' \cap T\rangle$. It was shown by Dyer ([19]) and Deodhar ([10]) independently that any reflection subgroup is also a Coxeter system; moreover, any reflection subgroup has a canonical set of Coxeter generators $\chi(W') = \{t \in T|\ N(t) \cap W' = \{t\}\}$. We say a reflection subgroup is *dihedral* if it is generated by two distinct reflections or equivalently $|\chi(W')| = 2$. We let $l_{(W',\chi(W'))} : W' \to \mathbb{N}$ be the length function for $(W', \chi(W'))$ and $T' = W' \cap T$ be the set of reflections of W'. Due to [19], $N_{(W',\chi(W'))}(x) = N(x) \cap W'$ for all reflection subgroups W' where $N_{(W',\chi(W'))}$ is the reflection cocycle for $(W', \chi(W'))$.

Any dihedral reflection subgroup, $W' = \langle s_\alpha, s_\beta\rangle$, is contained in a unique maximal dihedral reflection subgroup, namely $\langle s_\gamma|\ \gamma \in \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta)\rangle$. We let

$$\mathscr{M} = \{W' < W|\ W' \text{ is a maximal dihedral reflection subgroup}\}$$

$$\mathscr{M}_\infty = \{W' \in \mathscr{M}|\ |W'| = \infty\}.$$

For any reflection $t \in T$, we get the following set:

$$\mathscr{M}_t = \{W' \in \mathscr{M} |\ t \in W'\}.$$

The previous remarks imply that

$$T \setminus \{t\} = \dot{\bigcup}_{W' \in \mathscr{M}_t} ((W' \cap T) \setminus \{t\}) \tag{2.3.1}$$

where $\dot{\cup}$ represents a disjoint union.

Since any reflection subgroup is also a Coxeter system $(W', \chi(W'))$, we let $\Phi_{W'}$, $\Phi^+_{W'}$, $\Pi_{W'} \subseteq \Phi$ be the set of roots, positive roots, and simple roots for $(W', \chi(W'))$ sitting inside the root system for (W, S). We have that $\Phi_{W'} = \{\alpha \in \Phi |\ s_\alpha \in W'\}$.

For any $w \in W$, there exists a unique $x \in W'w$ with $N(x) \cap W' = \emptyset$ (likewise a unique $x \in wW'$ with $N(x^{-1}) \cap W' = \emptyset$). According to [19], we have $\chi(w^{-1}W'w) = x^{-1}\chi(W')x$. Let $w = yx$ with $y \in W'$. Then $\alpha \mapsto w^{-1}(\alpha) = x^{-1}y^{-1}(\alpha)$ induces a bijection $\Phi_{W'} \xrightarrow{\cong} \Phi_{w^{-1}W'w}$. This bijection is obtained by composing the bijection $\alpha \to y^{-1}(\alpha) : \Phi_{W'} \xrightarrow{\cong} \Phi_{W'}$ given by the action of $y^{-1} \in W'$ on the root system of W' and the bijection $\alpha \to x^{-1}(\alpha) : \Phi_{W'} \xrightarrow{\cong} \Phi_{x^{-1}W'x} = \Phi_{w^{-1}W'w}$ give by the action of x. Since $N(x) \cap W' = \emptyset$, the latter bijection preserves positive roots as well.

2.4 The Weak Order on T and the Set of Elementary Reflections

Much of the terminology of this section is described in terms of the root system in [2]. We will instead use terminology associated to the set of reflections when possible. For any reflection $t \in T$, we define the *standard height* to be $h(t) := h_{(W,S)}(t) = \frac{l(t)-1}{2}$.

At this point, we define a partial order on the set of reflections. For any

$t, t' \in T$, we set $t \to t'$ if there is some $s \in S$ (it is unique if it exists) such that $t' = sts$ and $l(t') = l(t) + 2$ (or equivalently $h(t') = h(t) + 1$). Let $\leq$ be the partial order on T given as the reflexive transitive closure of $\to$, and call $(T, \leq)$ the reflection poset or the weak order on T. It is clear that h is the rank function for $(T, \leq)$.

From the formula given in (2.3.1), we see that for $t \in T$ we have

$$h(t) = \sum_{W' \in \mathscr{M}_t} h_{(W', \chi(W'))}(t)$$

where $h_{(W', \chi(W'))}$ is the height function for $(W', \chi(W'))$. Now, we use this characterization of height to introduce the "∞-height" of a reflection as follows.

$$h^\infty(t) := \sum_{\substack{W' \in \mathscr{M}_t \\ |W'| = \infty}} h_{(W', \chi(W'))}(t). \tag{2.4.1}$$

We now introduce a special subset of the reflections. Let $T_0 := \{t \in T |\ h^\infty(t) = 0\}$. We call this set *the set of elementary reflections.* Using the bijection between T and Φ^+, this set is in bijective correspondence with the elementary roots defined in [5]. In that paper, Brink and Howlett also demonstrate that this set is finite. For another description of this fact, see [2].

More recently, Dyer has shown similar sets to be finite. More precisely, we define, for any $n \in \mathbb{N}$, $T_n := \{t \in T |\ h^\infty(t) = n\}$. For these sets, we have the following theorem, due to Dyer, found in [14].

Theorem 2.4.1. *Let (W, S) be a Coxeter system with reflections T. If S is a finite set, then T_n is a finite set for all $n \in \mathbb{N}$.*

These sets of reflections, as well as others like it, will be defined in more generality in Chapter 3, and we will make extensive use of these types of sets in Chapter 5 in order to demonstrate nice regularity properties of Coxeter systems. We will also use these sets to define certain modules for the generic Iwahori-Hecke algebra associated to (W, S) in Chapter 7.

2.5 Dominance Order on T

The set of elementary reflections also has a nice characterization in terms of another partial order on the set of reflections; it is the set of minimal elements of this partial order. Dominance order was initially defined in [5]. The elementary reflections and the dominance order are studied in more detail in [4]. The basic properties of dominance can be found in [2],[4], or [5], but we will include a few results here.

We will first define dominance on the root system associated to (W, S) and use the canonical bijection of positive roots with reflections to transport the definition to the reflections.

We say that $\alpha \in \Phi$ *dominates* $\beta \in \Phi$ written $\beta \preceq \alpha$ if, for all $w \in W$, $w(\alpha) \in \Phi^-$ implies that $w(\beta) \in \Phi^-$.

Lemma 2.5.1. *1. If $\alpha, \beta \in \Phi$, then α dominates β in Φ if and only if for some (respectively every) reflection subgroup W' of W with $s_\alpha, s_\beta \in W'$, α dominates β in $\Phi_{W'}$ (with respect to W').*

2. $\preceq$ is a partial order on Φ.

3. Multiplication by -1 is an order-reversing bijection of $(\Phi, \preceq)$ with itself.

4. The W-action on Φ is as a group of poset automorphisms of $(\Phi, \preceq)$.

Proof. First, suppose that $\beta, \alpha \in \Phi$ and that s_α and s_β are contained in a reflection subgroup W'. According to section 2.3, we have that for any $w \in W$, there is $x \in wW'$ with $N(x^{-1}) \cap W' = \emptyset$, and $\alpha \mapsto x(\alpha)$ is a bijection, which preserves positive roots, between $\Phi_{W'}$ and $\Phi_{wW'w^{-1}}$. Let $w = xy$ with $y \in W'$. Suppose that $\beta \preceq \alpha$ in $\Phi_{W'}$ with respect to W'. Then if $w(\alpha) \in \Phi^-$, we have $xy(\alpha) \in \Phi^-$ and so $y(\alpha) \in \Phi^-_{W'}$. Thus, $w(\beta) = xy(\beta) \in x(\Phi^-_{W'}) = \Phi^-_{W'} \subset \Phi^-$ and so $\beta \preceq \alpha$ in Φ. If $\beta \preceq \alpha$ in Φ, then by definition, $w(\alpha) \in \Phi^-$ implies $w(\beta) \in \Phi^-$ for all $w \in W$ and so this is true for all $w' \in W$ as required for 1. We now describe the dominance order on dihedral Coxeter systems. Let (W, S) be dihedral, with $S = \{r, s\}$. Then if W is finite, all the (finite) elements of Φ are incomparable in $\preceq$. If W is infinite, then the dominance order is given by the coarsest partial order satisfying the relations

$$\cdots - \alpha_{srsrs} \prec -\alpha_{srs} \prec -\alpha_s \prec \alpha_r \prec \alpha_{rsr} \prec \alpha_{rsrsr} \prec \cdots$$

and

$$\cdots - \alpha_{rsrsr} \prec -\alpha_{rsr} \prec -\alpha_r \prec \alpha_s \prec \alpha_{srs} \prec \alpha_{srsrs} \prec \cdots .$$

In either of these cases, it is clear that $(\Phi, \preceq)$ is a partial order. For general (not necessarily dihedral) (W, S), it is clear that $\preceq$ is reflexive and transitive by definition. Then, antisymmetry in general follows by using antisymmetry in a reduction to a dihedral reflection subgroup. Namely, suppose $\beta \preceq \alpha \preceq \beta$ with $\alpha \neq \beta$ in Φ, and then consider $W' = \langle s_\alpha, s_\beta \rangle$. Due to 1, we see that $\beta \preceq \alpha \preceq \beta$ in $\Phi_{W'}$, and by antisymmetry in the dihedral case, we get that $\alpha = \beta$, which is a contradiction. Finally, 3 and 4 are simple consequences of the definition of dominance. In particular if $\beta \preceq \alpha$ then $w(\beta) \preceq w(\alpha)$ since for any $u \in W$ we have

$uw(\alpha) \in \Phi^-$ implies that $uw(\beta) \in \Phi^-$. $\square$

Then, as stated before, we use the canonical bijection between Φ^+ and T to define the notion of dominance on the set of reflections. We have that $s_\beta \preceq s_\alpha$ if and only if $\beta \preceq \alpha$ in Φ^+. This is equivalent to $t' \preceq t$ if for all $w \in W$ we have $l(wt) < l(w)$ implies that $l(wt') < l(w)$; in this case, we say that t dominates t'. We will write $(T, \preceq)$ when referring to the set of reflections partially ordered by dominance.

We recall that in a poset, we say an element a covers an element b if $b < a$ and there is no $z \in P$ with $b < z < a$.

Lemma 2.5.2. *Let (W, S) be a Coxeter system.*

1. *If $s, t \in T_0$ generate an infinite subgroup, then $\{s, t\}$ is the set of canonical Coxeter generators for the maximal dihedral reflection subgroup of (W, S) containing $\{s, t\}$.*

2. *If $\alpha, \beta \in \Phi^+$ are distinct elementary roots (i.e. s_α and s_β are in T_0) and $\langle s_\alpha, s_\beta\rangle$ is an infinite dihedral reflection subgroup then β covers $-\alpha$ in dominance order.*

Proof. Due to the definition of h^∞ in equation (2.4.1), we know that $t \in T_0$ if and only if

$$0 = h^\infty(t) = \sum_{\substack{W' \in \mathscr{M}_t \\ |W'| = \infty}} h_{(W', \chi(W'))}(t)$$

which implies that $h_{(W', \chi(W'))}(t) = 0$ for all infinite dihedral reflection subgroups W'. Therefore, by the definition of the height, we see that $t \in \chi(W')$ for all infinite, maximal dihedral reflection subgroups of W with $t \in W' \cap T$. We see that 1 follows from this fact. To prove 2, since $\langle s_\alpha, s_\beta\rangle$ is infinite then $-\alpha \prec \beta$

by the description of dominance in infinite dihedral reflection subgroups given in 2.5.1. Suppose that $\gamma \in \Phi$ and $-\alpha \prec \gamma \prec \beta$. Since β is elementary, we cannot have $\gamma \in \Phi^+$; similarly, since α is elementary and $-\gamma \prec \alpha$ we cannot have $-\gamma \in \Phi^+$ so that $\gamma \notin \Phi^-$. This contradicts the fact that $\Phi = \Phi^+ \dot{\cup} \Phi^-$. □

Remark 2.5.3. In [14], Dyer uses the notion of the imaginary cone in a Coxeter system (see Chapter 11) to characterize dominance. He demonstrates that every covering $\alpha' \prec \beta'$ in dominance order is in the W-orbit of a covering $\alpha \prec \beta$ as in part 2 of 2.5.2.

CHAPTER 3

GENERALIZED LENGTH FUNCTIONS

In this chapter, we begin by introducing a family of generalized length functions on a Coxeter system. These length functions replace the standard length function on (W, S) by taking values in $\mathbb{N}^k$ for some $k \geq 1$ instead of simply taking values in $\mathbb{N}$. In a similar manner to section 2.4, these length functions will allow us to define corresponding height functions and, more importantly, a class of "infinite-height" functions on the set of reflections. These height functions lead to a partition of the set of reflections with interesting applications. Due to Theorem 2.4.1, these sets of reflections turn out to be finite. We will use the finiteness property to develop many canonical automata in Chapter 4.

We proceed by interpreting these height functions in terms of the poset $(T, \leq)$ (weak order) as defined in section 2.4. In turn, we further obtain monotonicity properties of these length and height functions with respect to the dominance order.

Finally, we demonstrate a basic recurrence formula for the reflection cocycle on (W, S) in terms of this new partition of T. We end with a few results that will be important in the following chapters. Many of the results generalize some of the results in [18].

3.1 Commutative Monoids

At this point we want to define a commutative monoid $\mathcal{M}$. We have two equivalent definitions for $\mathcal{M}$ which we introduce below. Both definitions will be useful for different applications, but it will be clear from context how we want to view $\mathcal{M}$ since one definition will have multiplicative notation and the other will have additive notation.

Let $\mathbf{X}$ be a finite set of indeterminates and $\mathbb{Z}[\mathbf{X}]$ be the corresponding integral polynomial ring. We say $\mathbf{m} \in \mathbb{Z}[\mathbf{X}]$ is a monomial if $\mathbf{m} = \prod_{x\in\mathbf{X}} x^{a_x}$ where $a_x \in \mathbb{N}$ for all $x \in \mathbf{X}$. Let $\mathcal{M} := \mathcal{M}_{\mathbf{X}}$ be the set of all monomials of $\mathbb{Z}[\mathbf{X}]$, that is $\mathcal{M} := \{\mathbf{m}|\ \mathbf{m} \in \mathbb{Z}[\mathbf{X}];\ \mathbf{m} \text{ is a monomial}\}$. For any $\mathbf{m} \in \mathcal{M}$, $\mathbf{m} = \prod_{x\in\mathbf{X}} x^{a_x}$, define the degree of $\mathbf{m}$ to be $\deg(\mathbf{m}) := \sum_{x\in\mathbf{X}} a_x$.

If we fix an order for $X = \{x_1, ..., x_k\}$, we can also consider the commutative monoid $\mathbb{N}^k$ defined by the map $\zeta : \mathcal{M} \to \mathbb{N}^k$ given by $\mathbf{m} = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \mapsto (a_1, ..., a_k)$. Since ζ is an isomorphism, we will abuse notation by setting $\mathbb{N}^k = \mathcal{M}$. The manner in which we view $\mathcal{M}$ will be clear depending on whether we use additive or multiplicative notation. We note that for $\mathbf{m} \in \mathcal{M}$, $\deg(\mathbf{m}) = \sum_{x\in X} a_x$ is independent of how we view $\mathcal{M}$. For any monomial $\mathbf{m} = (a_1, ..., a_k)$ we let $\mathbf{m}(i) := a_i$.

For $\mathbf{X}$ as above, let $\mathbf{X}^{-1} = \{x^{-1}|\ x \in \mathbf{X}\}$. We can define $\mathcal{M}' := \{\mathbf{m}|\ \mathbf{m} \in \mathbb{Z}[\mathbf{X}, \mathbf{X}^{-1}];\ \mathbf{m} \text{ is a monomial}\} \cong \mathbb{Z}^k$. This set has a natural group structure. We will also view $\mathcal{M}$ as a subset of $\mathcal{M}'$; in the case when we view $\mathcal{M}$ additively, this will explain using the notation $\mathbf{m} - \mathbf{n}$ (and will explain using division when viewing $\mathcal{M}$ multiplicatively).

We place a partial order $\leq_{\mathcal{M}}$ on $\mathcal{M}$ as follows. For any $\mathbf{m} \in \mathcal{M}$ and $\mathbf{n} \in \mathcal{M}$, we say $\mathbf{m} \leq_{\mathcal{M}} \mathbf{n}$ if and only if $\frac{\mathbf{n}}{\mathbf{m}} \in \mathcal{M} \subset \mathcal{M}'$ (additively this means $\mathbf{n} - \mathbf{m} \in$

$\mathcal{M} \subset \mathcal{M}')$. For $x \in \mathbf{X}$, we say that $\mathbf{m} \lhd_x \mathbf{n}$ if $\frac{\mathbf{n}}{\mathbf{m}} = x$ and call this a covering relation of type x. It is clear that if $\mathbf{m} \lhd_x \mathbf{n}$ and $\mathbf{m} \leq_{\mathcal{M}} \mathbf{m}' \leq_{\mathcal{M}} \mathbf{n}$ then $\mathbf{m}' = \mathbf{m}$ or $\mathbf{m}' = \mathbf{n}$. From this point on, we will write $\leq := \leq_{\mathcal{M}}$; this will not cause confusion with $\leq$ on T as the context will make clear which $\leq$ is intended.

Now, by an *ordered generalized partition* of a monomial $\mathbf{m}$ we will mean a tuple $\mathbf{M} = (\mathbf{m_1}, \mathbf{m_2}, ...)$, $\mathbf{m}_n = \mathbf{0}$ for $n \gg 0$, with $\deg(\mathbf{m_i}) \geq \deg(\mathbf{m_j})$ for all $i < j$ along with $\prod_i \mathbf{m_i} = \mathbf{m}$. Next, we define an equivalence relation, $\sim$, on the set of ordered generalized partitions in the following way. Let $(\mathbf{m}_1, \mathbf{m}_2, ...) \sim (\mathbf{n}_1, \mathbf{n}_2, ...)$ if there exists a permutation of the natural numbers, σ, such that $\mathbf{m}_i = \mathbf{n}_{\sigma(i)}$ for all i. For $(\mathbf{m}_1, \mathbf{m}_2, ...)$ an ordered partition, we let $\mathbf{M} = [(\mathbf{m}_1, \mathbf{m}_2, ...)]$ be the equivalence class of the ordered partition. We call these equivalence classes generated by this relation *unordered generalized partitions*. For any unordered partition $\mathbf{M} = [(\mathbf{m}_1, \mathbf{m}_2, ...)]$, we define $|\mathbf{M}| := \prod_i \mathbf{m}_i$. All the terminology of the partial order on $\mathcal{M}$ can be phrased additively if necessary.

At this point, we can define a partial order on the set of unordered generalized partitions. Let $\mathbf{M}$ and $\mathbf{N}$ be any two generalized partitions. Then we say that $\mathbf{M} \leq \mathbf{N}$ if there exists $(\mathbf{m}_1, \mathbf{m}_2, ...) \in \mathbf{M}$ and $(\mathbf{n}_1, \mathbf{n}_2, ...) \in \mathbf{N}$ such that $\mathbf{m}_i \leq \mathbf{n}_i$ in $\mathcal{M}$ for all i. For any $x \in \mathbf{X}$, we define the covering relation of type x by $\mathbf{M} \lhd_x \mathbf{N}$ if there exists $(\mathbf{m}_1, \mathbf{m}_2, ...) \in \mathbf{M}$ and $(\mathbf{n}_1, \mathbf{n}_2, ...) \in \mathbf{N}$ and $j \in \mathbb{N}$ such that $\mathbf{m}_i = \mathbf{n}_i$ for all $i \neq j$ and $\mathbf{m}_j \lhd_x \mathbf{n}_j$ in $\mathcal{M}$.

3.2 A General Length for Coxeter Systems

Suppose we have a fixed Coxeter system (W, S). Let $\mathcal{M}$ be the additive, commutative monoid described in section 3.1 where $\mathcal{M}$ is generated by a finite set $\mathbf{X} = \{x_1, ..., x_k\}$ of indeterminates. Suppose that we are given a surjective function $p : T \to \mathbf{X}$ satisfying $p(t) = p(t')$ if t is conjugate to t', i.e. if there exists $v \in W$ with $t = vt'v^{-1}$. For any reflection $t \in T$, we say that t is of *type* x_i if $p(t) = x_i$ where $x_i \in \mathbf{X}$ (for simplicity we will usually say that t is of type $p(t)$). Until further notice, we will view $\mathcal{M}$ with additive notation and we will, for ease, let $p(t)$ represent $\zeta(p(t))$.

We define a family of "length" functions in the following way. Let W' be any reflection subgroup of W and $w \in W'$ where $w = r_1 ... r_n$ is a reduced expression with $r_i \in \chi(W')$. Then we define $l_{p,W'} : W' \to \mathcal{M}$ by $l_{p,W'}(w) = l_{p,W'}(r_1...r_n) = p(r_1) + p(r_2) + \cdots + p(r_n)$. See figure 3.1 for an example.

Similarly, for any reflection $r \in W' \cap T$ with $r = r_1...r_m r_{m+1}...r_{2m+1}$, we define a corresponding "height function" $h_{p,W'} : T \to \mathcal{M}$ by $h_{p,W'}(r) = p(r_1) + p(r_2) + \cdots + p(r_m)$. Finally, for any $r \in W' \cap T$ where $r = r_1...r_m r_{m+1}...r_{2m+1}$, we define $c(r) := p(r) = p(r_{m+1})$, which is called the *center* of r. Note that for $t \in T$, $l_{p,W'}(t) = c(t) + 2h_{p,W'}(t)$.

From these height functions, we get the following functions on reflections as well. For any union $\mathscr{O} \subset \mathscr{M}$ of W-orbits of maximal dihedral reflection subgroups on $\mathscr{M}$, we define

$$h_{W,p,\mathscr{O}} : t \mapsto \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{O}}} h_{p,W'}(t) \tag{3.2.1}$$

and

$$l_{W,p,\mathscr{O}} : t \mapsto c(t) + 2h_{W,p,\mathscr{O}}(t). \tag{3.2.2}$$

In this thesis, we will only consider the case where $\mathscr{O} = \mathscr{M}_\infty$.

Remark 3.2.1. We note that, considering terminology from section 2.4, we have $h^\infty(t) = \deg(h_{W,p,\mathscr{M}_\infty}(t))$. Also, if $\mathcal{M} = \mathbb{N}$ and $q(t) = 1$ for all $t \in T$, then

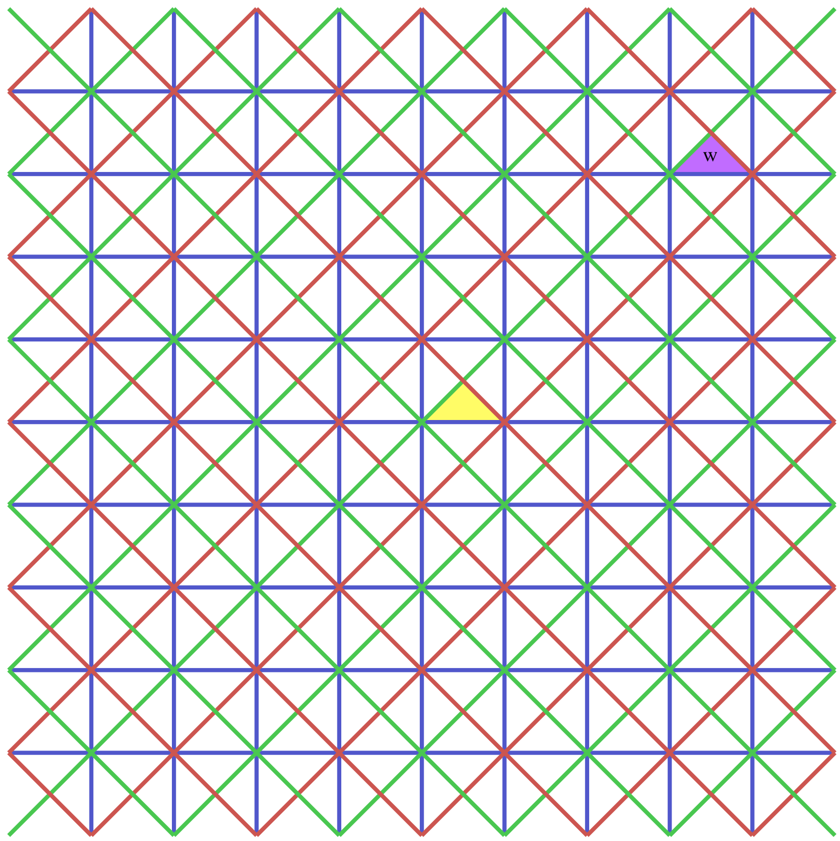


Figure 3.1. This diagram shows the idea of generalized length in $\widetilde{B}_2 = \langle r, s, t \mid r^2 = s^2 = t^2 = (rt)^2 = 1, (rs)^4 = (st)^4 = 1\rangle$. Each group element corresponds to one of the alcoves, with the yellow triangle representing the identity. Each reflection corresponds to one of the affine hyperplanes, and the color of the hyperplane distinguishes the conjugacy class. There are three conjugacy classes. The standard length counts the number of affine hyperplanes separating w from the yellow alcove. The generalized length counts this same number but retains additional information about the conjugacy class of each affine hyperplane separating w from the identity. For the w specified above we have $l(w) = 12$ while $l_p(w) = (3, 6, 3)$ where p represents the function taking distinct values on each conjugacy class.

$h^{\infty}(t) = h_{W,q,\mathscr{M}_\infty}(t)$.

Lemma 3.2.2. *Let (W, S) be a Coxeter system, and let $\leq_\emptyset$ be ordinary Bruhat order on (W, S) (see Chapter 8). Let $x, y \in W$ with $x \leq_\emptyset y$. Then for all reflection subgroups $W' \subseteq W$ with $x, y \in W'$, the following hold.*

1. $l_{p,W'}(x) \leq l_{p,W'}(y)$, *and*

2. $h_{p,W'}(x) \not> h_{p,W'}(y)$ *if* $x, y \in T \cap W'$.

3. $h_{p,W'}(x) \leq h_{p,W'}(y)$ *if* $x, y \in T \cap W'$ *and* $c(x) = c(y)$.

Proof. We know that $x \leq_\emptyset y$ in W if and only if $x \leq_\emptyset y$ in W' for all $x, y \in W' \subseteq W$. The subword characterization of Bruhat order and the definition of $l_{p,W}$ in terms of reduced expressions imply 1. Finally, 2 and 3 follow from 1. □

3.3 Infinite Heights and Generalized Partitions for Reflections

According to equation (3.2.1) we can attach an unordered generalized partition to each reflection t, $\mathbf{M}_{W,p,\mathscr{M}_\infty}(t)$, where each part, $\mathbf{m}_i$, is given by some distinct $h_{p,W'}(t)$ with $W' \in \mathscr{M}_\infty$.

Lemma 3.3.1. *Let $t \in T$ and $s \in S$ with $l_p(sts) = l_p(t) + 2p(s)$. Then*

1. $h_{W,p,\mathscr{M}_\infty}(sts) = h_{W,p,\mathscr{M}_\infty}(t)$ *if $\langle s, t\rangle$ is finite and* $h_{W,p,\mathscr{M}_\infty}(sts) = h_{W,p,\mathscr{M}_\infty}(t) + p(s)$ *if $\langle s, t\rangle$ is infinite.*

2. $\mathbf{M}_{W,p,\mathscr{M}_\infty}(sts) = \mathbf{M}_{W,p,\mathscr{M}_\infty}(t)$ *if $\langle s, t\rangle$ is finite. If $\langle s, t\rangle$ is infinite then* $\mathbf{M}_{W,p,\mathscr{M}_\infty}(t) \lhd_{p(s)} \mathbf{M}_{W,p,\mathscr{M}_\infty}(sts)$.

Proof. Let W' be a maximal dihedral reflection subgroup. According to 2.3 if $s \notin W'$, then $N(s) \cap W' = \emptyset$ so $(W', \chi(W')) \cong (sW's, s\chi(W')s)$ and thus

$h_{p,sW's}(sts) = h_{p,W'}(t)$. If $s \in W'$, then $s \in \chi(W')$ and $sW's = W$, and it follows that $h_{p,W'}(sts) = h_{p,W'}(t) + p(s)$. Now, since $\mathscr{M}_{sts} = \{sW's \mid W' \in \mathscr{M}_t\}$, if $\langle s,t\rangle \in \mathscr{M}_t \cap \mathscr{M}_\infty$ then we have

$$\begin{aligned} h_{W,p,\mathscr{M}_\infty}(sts) &= \sum_{\substack{W' \in \mathscr{M}_{sts} \\ W' \in \mathscr{M}_\infty}} h_{p,W'}(sts) = \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} h_{p,sW's}(sts) \\ &= p(s) + \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} h_{p,W'}(t) = p(s) + h_{W,p,\mathscr{M}_\infty}(t). \end{aligned}$$

Otherwise we have

$$\begin{aligned} h_{W,p,\mathscr{M}_\infty}(sts) &= \sum_{\substack{W' \in \mathscr{M}_{sts} \\ W' \in \mathscr{M}_\infty}} h_{p,W'}(sts) = \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} h_{p,sW's}(sts) \\ &= \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} h_{p,W'}(t) = h_{W,p,\mathscr{M}_\infty}(t). \end{aligned}$$

Based on these results, the lemma follows clearly. □

3.4 Infinite Height and the Weak Order

Consider the Hasse diagram of the weak order $(T, \leq)$ as a directed graph with vertex set T and directed edges (t,t') from t to t' for all $t,t' \in T$ with $t \to t'$. Furthermore, we can regard this graph as a labeled directed graph by labeling the edges as follows. For any edge (t,t'), we know there is a unique $s \in S$ with $t' = sts$. From this, we can label the edge (t,t') with label $p(s)$, and we call this edge an edge of type $p(s)$. We say that an edge (t,t') is $p(s)$-short if it is labeled with $p(s)$ and if $h_{W,p,\mathscr{M}_\infty}(t') = h_{W,p,\mathscr{M}_\infty}(t)$, i.e. if $\langle t,s\rangle$ is finite. We say that the edge is $p(s)$-long if it is labeled with $p(s)$ and $h_{W,p,\mathscr{M}_\infty}(t') = h_{W,p,\mathscr{M}_\infty}(t) + p(s)$, i.e. if $\langle t,s\rangle$ is infinite. Based on this, we see that for any $t \leq t'$ in $(T, \leq)$, the number of $p(s)$-long edges (x_{i-1}, x_i) in a chain $t = x_0 \to x_1 \to ... \to x_n = t'$ is the $\mathbf{m}(j) = h_{W,p,\mathscr{M}_\infty}(t')(j) - h_{W,p,\mathscr{M}_\infty}(t)(j)$ where $p(s) = x_j$. Furthermore, for any fixed t, then maximal number of $p(s)$-long edges in chains $x_0 \to x_1 \to ... \to x_n = t$ is $h_{W,p,\mathscr{M}_\infty}(t)(j)$ and this maximum is realized if $x_0 \in S$.

Corollary 3.4.1. *The map $t \mapsto \mathbf{M}_{W,p,\mathscr{M}_\infty}(t)$ is an order preserving map from the reflection poset of (W, S) to the poset of generalized partitions.*

Proof. This follows directly from part 2 of Lemma 3.3.1. □

Remark 3.4.2. Notice that for $t \leq t'$, we have that $c(t) = c(t')$ by definition of the weak order.

3.5 The Root Poset

The reflections endowed with weak order as a poset is isomorphic to a subposet of another poset which we briefly introduce. We call this poset the root poset denoted $(\Phi, \leq)$. Define $\to$ on Φ by $\alpha \to \beta$ if and only if there is some $\gamma \in \Pi$ such that $\beta = s_\gamma(\alpha)$ and $(\gamma|\alpha) < 0$, i.e. $\beta - \alpha \in \mathbb{R}_{>0}\gamma$. Then, we let $\leq$ be the reflexive transitive closure of $\to$.

We can consider the Hasse diagram of $(\Phi, \leq)$ as a directed graph with vertex set Φ and edges (α, β) whenever $\alpha \to \beta$. We label all edges by their type $p(s_\gamma)$ where s_γ is uniquely determined by $\beta = s_\gamma(\alpha)$ where $\gamma \in \Pi$. We say an edge is $p(s_\gamma)$-long if it is labeled with $p(s_\gamma)$ and if $(\alpha|\gamma) \leq -1$, otherwise we call it $p(s_\gamma)$-short. It is clear that the relations $(\Phi^+, \to)$ and $(\Phi^+, \leq)$ are obtained by transporting the relations from $(T, \to)$ and $(T, \leq)$ by means of the bijection $s_\alpha \mapsto \alpha$ from T to Φ^+. We note that this correspondence also respects edge lengths and types, i.e. the edge (α, β) is $p(s)$-long if and only if the corresponding edge (s_α, s_β) is $p(s)$-long.

Using this correspondence, we translate the basic properties of the reflection poset into corresponding properties of the root poset. In particular, we introduce the following length function on the roots.

$$L_{W,p,\mathscr{M}_\infty}(\alpha) = \begin{cases} l_{W,p,\mathscr{M}_\infty}(s_\alpha) & \text{if } \alpha \in \Phi^+ \\ -l_{W,p,\mathscr{M}_\infty}(s_\alpha) & \text{if } \alpha \in \Phi^- \end{cases} \tag{3.5.1}$$

Lemma 3.5.1. *1. For $\alpha \leq \beta$ in Φ, the interval $[\alpha, \beta]$ is finite.*

2. The map $\alpha \mapsto -\alpha$ is an order-reversing bijection of $(\Phi, \leq)$ with itself.

3. Let $\alpha = \alpha_0 \to \alpha_1 \to \ldots \to \alpha_n = \beta$ be a maximal chain in $(\Phi, \leq)$. Then for any $x_i \in \mathbf{X}$, the number of x_i-long edges (α_{i-1}, α_i) in the chain (regarded as a path from α to β in the Hasse diagram) is equal to $\mathbf{m}(i)$ where $\mathbf{m} = \frac{L_{W,p,\mathscr{M}_\infty}(\beta) - L_{W,p,\mathscr{M}_\infty}(\alpha)}{2}$.

Proof. First, 1 follows from the corresponding fact that intervals are finite in $(T, \leq)$, and we remark that 2 is clearly true since $-\alpha \to \alpha$ for all $\alpha \in \Pi$.

Now, suppose that $\alpha, \beta \in \Phi^+$, then $s_\alpha \leq s_\beta$ in $(T, \leq)$. Let $d := s_\alpha \leq s_{\alpha_1} \leq \cdots \leq s_{\alpha_{n-1}} \leq s_\beta$ be the maximal chain from s_α to s_β. Then according to 3.4, $h_{W,p,\mathscr{M}_\infty}(s_\beta)(i) - h_{W,p,\mathscr{M}_\infty}(s_\alpha)(i)$ is the number of x_i-long edges in the chain. By definition of $l_{W,p,\mathscr{M}_\infty}$ we have that

$$h_{W,p,\mathscr{M}_\infty}(s_\beta) - h_{W,p,\mathscr{M}_\infty}(s_\alpha) = \frac{l_{W,p,\mathscr{M}_\infty}(s_\beta) - c(s_\beta)}{2} - \left(\frac{l_{W,p,\mathscr{M}_\infty}(s_\alpha) - c(s_\alpha)}{2}\right)$$

and by Remark 3.4.2 we know that since $s_\alpha \leq s_\beta$ then $c(s_\alpha) = c(s_\beta)$ and the result follows in this case. Similarly, if $\alpha, \beta \in \Phi^-$ then $-\alpha, -\beta \in \Phi^+$ and $-\beta \leq -\alpha$ and the proof follows from above.

This leaves us with the case that $\alpha \in \Phi^-$ and $\beta \in \Phi^+$ (notice we cannot have

$\alpha \in \Phi^+$ and $\beta \in \Phi^-$). Let $\alpha_i \in \Phi^-$ and $\alpha_{i+1} \in \Phi^+$, then since $\alpha_i \to \alpha_{i+1}$ we must have $-\alpha_i = \alpha_{i+1}$ (since the only positive root made negative by s_γ is γ for $\gamma \in \Pi$). In this case, we have that the edge (α_i, α_{i+1}) is $p(s_{\alpha_{i+1}})$-long. Note, that since $\alpha \leq \alpha_{i+1} \leq \beta$ we have $c(s_\alpha) = c(s_\beta) = c(s_{\alpha_{i+1}}) = p(s_{\alpha_{i+1}})$ by Remark 3.4.2 and the definition of c. Then, using the previous argument, we know that the number of x_i-long edges in $\alpha \to \alpha_1 \to \cdots \to \alpha_i$ is equal to the number of x_i-long edges in $-\alpha_i \to -\alpha_{i-1} \to \cdots \to -\alpha$ which is $h_{W,p\mathscr{M}_\infty}(s_\alpha)(i)$. Also, the number of x_i-long edges in $\alpha_{i+1} \to \alpha_{i+2} \to \cdots \to \beta$ is $h_{W,p,\mathscr{M}_\infty}(s_\beta)(i)$. Now we have the total number of x_i long edges is given by $\mathbf{m}(i)$ where

$$\begin{aligned}\mathbf{m} :&= h_{W,p\mathscr{M}_\infty}(s_\beta) + h_{W,p\mathscr{M}_\infty}(s_\alpha) + c(s_{\alpha_{i+1}}) \\ &= \frac{l_{W,p,\mathscr{M}_\infty}(s_\beta) - c(s_\beta)}{2} + \frac{l_{W,p,\mathscr{M}_\infty}(s_\alpha) - c(s_\alpha)}{2} + c(s_{\alpha_{i+1}}) \\ &= \frac{l_{W,p,\mathscr{M}_\infty}(s_\alpha) + l_{W,p,\mathscr{M}_\infty}(s_\beta)}{2}.\end{aligned}$$

□

3.6 Monotonicity Properties in Dominance Order on T

Recall that using the canonical bijection between Φ^+ and T, given by $\alpha \mapsto s_\alpha$, we transport the dominance order on Φ^+ to dominance order on T. We have that $s_\beta \preceq s_\alpha$ if and only if $\beta \preceq \alpha$. This is equivalent to $t' \preceq t$ if for all $w \in W$ we have $l(wt) < l(w)$ implies that $l(wt') < l(w)$; in this case, we say that t dominates t'.

Lemma 3.6.1. *Let $r, t \in T$.*

(a) t dominates r in T if and only if for some (respectively every) reflection group W' of W with $t, r \in W'$, t dominates r in dominance order on the set $W' \cap T$ of reflections of $(W', \chi(W'))$.

(b) t dominates r if and only if either (1) $t = r$, or (2) $\langle t, r\rangle$ is infinite and $h_{p,W}(rtr) < h_{p,W}(t)$.

(c) t dominates r if and only if either (1) $t = r$, or (2) $\langle t, r\rangle$ is infinite and $l_{p,W}(rt) < l_{p,W}(t)$ and $h_{p,W}(r) \not> h_{p,W}(t)$.

(d) If $r \preceq t$ then $t = r$ if and only if $h_{p,W}(r) = h_{p,W}(t)$.

(e) Let $t \in T$, $x_i \in \mathbf{X}$, and $\mathbf{m} = h_{W,p,\mathscr{M}_\infty}(t)$. Then, $\mathbf{m}(i) = |\{t' \in T | t' \prec t;\ p(t') = x_i\}|$.

(f) If $r \preceq t$, then $h_{W,p,\mathscr{M}_\infty}(r) \leq h_{W,p\mathscr{M}_\infty}(t)$.

Proof. First of all, (a) follows from part 1 of Lemma 2.5.1 via the bijection between positive roots and T. We simply transfer the structure of dominance by this bijection and the result holds.

Next, (e) is clear for finite dihedral groups. Now, suppose we have an infinite dihedral group (W, S) with $S = \{r, s\}$. Then, as in 2.5.1, we know that the dominance order on T in this case is the coarsest partial order satisfying

$$s \prec srs \prec srsrs \prec srsrsrs \prec \cdots$$

and

$$r \prec rsr \prec rsrsr \prec rsrsrsr \prec \cdots .$$

Using this description, (e) is readily checked. Then, for general (W, S), for any

$x_i \in \mathbf{X}$ we have

$$\begin{aligned}
h_{W,p,\mathscr{M}\infty}(t)(i) &= \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} h_{p,W'}(t)(i) \\
&= \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} |\{t' \in W' \cap T|\ t' \prec_{W',\chi(W')} t;\ p(t') = x_i\}| \\
&= \sum_{\substack{W' \in \mathscr{M}_t \\ W' \in \mathscr{M}_\infty}} |\ t' \in W' \cap T|\ t' \prec t;\ p(t') = x_i\}| \\
&= |\{t' \in T|\ t' \prec t';\ p(t') = x_i\}|
\end{aligned}$$

where the first equality is the definition, the second equality follows from the dihedral case, the third equality follows from (a), and the final equality follows from equation (2.3.1).

Now, we can also prove (b) and (c) using this reduction to rank 2 as well. We note that by the description for dihedral reflection groups above, (b) and (c) hold by inspection. For the general case of (b) with arbitrary (W, S), suppose that $t, r \in T$ with $r \preceq t$ and $t \neq r$. Let $W' = \langle r, t \rangle$. Since $r \preceq t$ in W', according to [4] we have that W' is infinite. The dihedral case shows that we have $h_{p,W'}(rtr) < h_{p,W'}(t)$. This implies that $rtr <_\emptyset t$ and we know $c(t) = c(rtr)$. Therefore, Lemma 3.2.2 implies that $h_{p,W}(rtr) < h_{p,W}(t)$. Conversely, if $t \neq r$ and $W' = \langle t, r \rangle$ is infinite with $h_{p,W}(rtr) < h_{p,W}(t)$, then $rtr <_\emptyset t$ so that $h_{p,W'}(rtr) \leq h_{p,W'}(t)$ by Lemma 3.2.2; from (b) in the dihedral case along with (a), we have $r \preceq t$. Next, (c) follows in a similar manner. For $r \preceq t \in T$ with $t \neq r$, let $W' = \langle r, t \rangle$. Then by the dihedral case, $l_{p,W'}(rt) < l_{p,W'}(t)$ and $h_{p,W'}(r) \ngtr h_{p,W'}(t)$. Hence $rt <_\emptyset t$, and $r <_\emptyset t$ (by height requirement) in W' and thus in W as well. Therefore, by Lemma 3.2.2, $l_{p,W}(rt) < l_{p,W}(t)$ and $h_{p,W}(r) \ngtr h_{p,W}(t)$. Conversely, if $W' = \langle r, t \rangle$ infinite and $l_{p,W}(rt) < l_{p,W}(t)$ then $rt \leq_\emptyset t$ and so $l_{p,W'}(rt) < l_{p,W'}(t)$. This implies

$r \leq_\emptyset t$ or $t \leq_\emptyset r$ (by inspection of infinite dihedral case). Due to the fact that $h_{p,W}(r) \not> h_{p,W}(t)$, we must have $r \leq_\emptyset t$ and so $h_{p,W'}(r) \not> h_{p,W'}(t)$ by Lemma 3.2.2; consequently, $r \preceq t$ in W' and thus in W by (a). Finally, (d) follows from (b) and (f) follows from (e) directly. □

3.7 A Partition of the Reflections

At this point, we define certain subsets of the set of reflections, T, of a given Coxeter group (W, S). These sets are analogs of T_n from section 2.4. For any monomial $\mathbf{m} \in \mathcal{M}$, define $T_{\mathbf{m}} := \{t \in T | \ h_{W,p,\mathscr{M}_\infty}(t) = \mathbf{m}\}$.

Proposition 3.7.1. *Suppose S is finite. Then $T_{\mathbf{m}}$ is finite for any $\mathbf{m} \in \mathcal{M}$.*

Proof. This follows from Theorem 2.4.1 since $T_{\mathbf{m}} \subset T_{\deg(\mathbf{m})}$ and $\deg(\mathbf{m}) \in \mathbb{N}$. □

Recall $(\mathcal{M}, \leq)$ from section 3.1. For any $\mathbf{m} \in \mathcal{M}$, let $T_{\leq \mathbf{m}} := \bigcup_{\mathbf{n} \leq \mathbf{m}} T_{\mathbf{n}}$.

3.8 Recurrence Formulae

We now further extend the results of [18]. Recall the reflection cocycle $N : W \to \mathscr{P}(T)$. For any $\mathbf{m} \in \mathcal{M}$ and $w \in W$, we define $N_{\mathbf{m}}(w) := N(w) \cap T_{\leq \mathbf{m}}$.

Proposition 3.8.1. *Let $w \in W$, $s \in S$, and $\mathbf{m} \in \mathcal{M}$.*

(a) If $s \notin N_{\mathbf{m}}(w)$ then $N_{\mathbf{m}}(sw) = (\{s\} \cup sN_{\mathbf{m}}(w)s) \cap T_{\leq \mathbf{m}}$.

(b) If $s \in N_{\mathbf{m}}(w)$ and $\mathbf{m} \geq p(s)$, then $N_{\mathbf{m}-p(s)}(sw) = (sN_{\mathbf{m}}(w)s \setminus \{s\}) \cap T_{\leq \mathbf{m}-p(s)}$

Proof. Recall that $N(sw) = \{s\} + sN(w)s$ so the containment of the right hand side into the left hand side is clear for both parts. Now assume that $s \notin N_{\mathbf{m}}(w)$.

Then $s \notin N(w)$. Let $t \in N_{\mathbf{m}}(sw) = N(sw) \cap T_{\leq \mathbf{m}} = (\{s\} \cup sN(w)s) \cap T_{\leq \mathbf{m}}$, and then let $t' = sts$. If $t = s$ then we clearly have $t = s \in (\{s\} \cup sN_{\mathbf{m}}(s)s) \cap T_{\leq \mathbf{m}}$. If $t \neq s$, then $t \in sN(w)s$ and so $t' \in N(w)$. If $t = t'$ then we already have assumed $t' = t \in T_{\leq \mathbf{m}}$ and so this proves that $t' \in N_{\mathbf{m}}(w)$ so that $t \in sN_{\mathbf{m}}(w)s$ and thus t is in the right hand side. If $\langle s, t \rangle$ is finite or if $l_p(t) = l_p(t') + 2p(s)$ then by Lemma 3.3.1 we know that $h_{W,p,\mathscr{M}_\infty}(t') \leq h_{W,p,\mathscr{M}_\infty}(t) \leq \mathbf{m}$ by assumption that $t \in T_{\leq \mathbf{m}}$; thus, $t' \in N_{\mathbf{m}}(w)$ and $t \in sN_{\mathbf{m}}(w)s$ as well as the right hand side. This only leaves the case where $\langle s, t \rangle$ is infinite and $l_p(t') = l_p(t) + 2p(s)$. Let $W' = \langle t, s \rangle$. Since $s \in S$, then we know that $\chi(W') = \{r, s\}$ for some $r \in T$. By assumption, $s \notin N(t)$ and $t \neq 1 \in W'$ so we must have $r \in N(t)$. Therefore, we know by the proof of Lemma 2.5 that t dominates r. We have assumed that $t \in N(sw)$ and the dominance then implies that $r \in N(sw)$ as well. Also, by our first assumption, we know that $s \in N(sw)$ as well, but this forms a contradiction since we now have $\langle t, s \rangle = \langle r, s \rangle \subset N(sw)$ (see [17]). However $N(sw)$ is finite, and we know that $\langle t, s \rangle$ is infinite. Thus, we have a contradiction proving (a).

To prove part (b), assume that $s \in N_{\mathbf{m}}(w)$. Thus, we have that $s \in N(w)$ and so by symmetric difference we must have $N(sw) = sN(w)s \setminus \{s\}$. Let $t \in N_{\mathbf{m}-p(s)}(sw) = (sN(w)s \setminus \{s\}) \cap T_{\leq \mathbf{m}-p(s)}$. Set $t' := sts$. Clearly we have that $h_{W,p,\mathscr{M}_\infty}(t') \leq \mathbf{m} - p(s) + p(s) = \mathbf{m}$ so $t' \in T_{\leq \mathbf{m}}$. Also, $t' \in N(w)$ by assumption that $t \in sN(w)s \setminus \{s\}$. Thus, $t' \in N_{\mathbf{m}}(w)$ and so $t = st's \in sN_{\mathbf{m}}(w)s \setminus \{s\}$, and we already have assumed that $t \in T_{\leq \mathbf{m}-p(s)}$. □

3.9 Monoids with Poset Structure

Let $(P, \leq)$ be any poset and let $\mathcal{M}$ be an additively written commutative monoid. Suppose we have a partial order $\leq_{\mathcal{M}}$ on $\mathcal{M}$ such that $\mathbf{0} \leq_{\mathcal{M}} \mathbf{m}$ for all

$\mathbf{m} \in \mathcal{M}$ and $\mathbf{m} \leq \mathbf{n}$ if and only if $\mathbf{m}+\mathbf{k} \leq \mathbf{n}+\mathbf{k}$ for all $\mathbf{m}, \mathbf{n}, \mathbf{k} \in \mathcal{M}$. In this case, we say $\leq_{\mathcal{M}}$ is compatible with the monoid structure on $\mathcal{M}$. Furthermore, let $q : P \to \mathcal{M} \setminus \{\mathbf{0}\}$ be a map. Define an operator $|-| : \mathscr{P}_{\text{fin}}(P) \to \mathcal{M}$ in the following way, where $\mathscr{P}_{\text{fin}}(P)$ represents all of the finite subsets of P. For finite $A \subset P$, we have

$$|A| := \sum_{a \in A} q(a) \tag{3.9.1}$$

Remark 3.9.1. Notice that if $\mathcal{M} = \mathbb{N}$ with the usual total order and $q(x) = 1$ for all $x \in P$ then this is the usual cardinality of a finite set.

An ideal I of P is a subset of P satisfying the following property: if $x \in I$ and $y \leq x$ in P then $y \in I$ as well. Any subset $X \subset P$ is contained in a smallest (under inclusion) ideal of P called the ideal generated by X. We say an ideal, I, is principal if I is generated by a single element $\{x\}$ with $x \in P$. We say that P is lower finite if every principal ideal is finite. If we have a lower finite poset, we define $h' : P \to \mathcal{M}$ by $h'(x) = |\{z \in P|\ z < x\}|$. For any $\Gamma \subset \mathcal{M}$ define $P_\Gamma := \{z \in P|\ h'(z) \in \Gamma\}$. We will consider certain cases like $\Gamma = \leq_{\mathcal{M}} \mathbf{m} := \{\mathbf{n} \in \mathcal{M}|\ \mathbf{n} \leq_{\mathcal{M}} \mathbf{m}\}$ and $\Gamma = <_{\mathcal{M}} \mathbf{m} := \{\mathbf{n} \in \mathcal{M}|\ \mathbf{n} <_{\mathcal{M}} \mathbf{m}\}$

Lemma 3.9.2. *Let P be a lower finite poset, I be a finite ideal of P and $\mathbf{m} \in \mathcal{M}$. Then*

(a) $|I| \leq_{\mathcal{M}} \mathbf{m}$ *iff* $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} \mathbf{m}$.

(b) $|I| = \mathbf{m}$ *iff* $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| = \mathbf{m}$ *and* $I \cap P_{<_{\mathcal{M}}\mathbf{m}} = I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$.

Proof. To prove (a), we will instead prove

$$|I| \not\leq_{\mathcal{M}} \mathbf{m} \text{ iff } |I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \not\leq_{\mathcal{M}} \mathbf{m}.$$

The fact that the right hand side implies the left is clear. So assume that $|I| \not\leq_{\mathcal{M}} \mathbf{m}$. Now, we define $\mathcal{B} := \{I' \subseteq I |\ I' \text{ is an ideal; } |I'| \not\leq_{\mathcal{M}} \mathbf{m}\}$. Since I is finite, $\mathcal{B}$ is clearly finite and nonempty since $I \subset \mathcal{B}$. Let $I_0 \in \mathcal{B}$ be a minimal element of $\mathcal{B}$ (with respect to inclusion). We note that $|\emptyset| = \mathbf{0}$ and due to the fact that $\mathbf{m} \geq 0$ for all $\mathbf{m} \in \mathcal{M}$, we must have $|I_0| \neq \emptyset$. Then, for any maximal element $t \in I_0$, we know that $J := I_0 \setminus \{t\}$ is an ideal satisfying $J \subsetneq I_0 \subset I$. So by the minimality of I_0, we must have $|J| \leq_{\mathcal{M}} \mathbf{m}$. Similarly, let $K = \{x \in I_0 |\ \{y \in I_0 |\ y > x\} = \emptyset\}$ be the set of maximal elements in I_0, then $J' = I_0 \setminus K$ must also have $|J'| \leq_{\mathcal{M}} \mathbf{m}$.

So, for any $t \in I_0$ we must have that $h'(t) = |\{s \in P |\ s < t\}| \leq_{\mathcal{M}} |I_0 \setminus K| = |J'| \leq_{\mathcal{M}} \mathbf{m}$. Therefore, $t \in P_{\leq_{\mathcal{M}}\mathbf{m}}$. So we have shown that $I_0 \subset I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$ and thus $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \geq_{\mathcal{M}} |I_0|$. If $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} \mathbf{m}$, then this would contradict the fact that $|I_0| \not\leq_{\mathcal{M}} \mathbf{m}$; thus, we have shown (a).

To show (b), we will use (a). First, suppose that $|I| = \mathbf{m}$. This implies that $|I| \not\leq_{\mathcal{M}} \mathbf{n}$ for all $\mathbf{n} \lhd_{\mathcal{M}} \mathbf{m}$ (where $\lhd_{\mathcal{M}}$ represents the covering relation for $\leq_{\mathcal{M}}$). Using part (a), we have that $|I \cap P_{\leq_{\mathcal{M}}\mathbf{n}}| \not\leq_{\mathcal{M}} \mathbf{n}$ for all $\mathbf{n} \lhd_{\mathcal{M}} \mathbf{m}$. Putting these all together we get that $|I \cap \cup_{\mathbf{k} \lhd_{\mathcal{M}} \mathbf{m}} P_{\leq_{\mathcal{M}}\mathbf{k}}| \not\leq_{\mathcal{M}} \mathbf{n}$ for all $\mathbf{n} \lhd_{\mathcal{M}} \mathbf{m}$. Thus, $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| \not\leq_{\mathcal{M}} \mathbf{n}$ for all $\mathbf{n} \lhd_{\mathcal{M}} \mathbf{m}$. However, we know that $I \cap P_{<_{\mathcal{M}}\mathbf{m}} \subseteq I$ so that $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} |I| = \mathbf{m}$, and thus we must have $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| = \mathbf{m}$. Now, since $|I| = \mathbf{m}$, we have also by (a) that $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} \mathbf{m}$. Then, by our previous work we know that $\mathbf{m} = |I \cap P_{<_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} |I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \leq_{\mathcal{M}} \mathbf{m}$ since $I \cap P_{<_{\mathcal{M}}\mathbf{m}} \subseteq I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$. Since $\leq_{\mathcal{M}}$ is compatible with the monoid structure, this implies that $|I \cap P_{=\mathbf{m}}| = 0$, and the fact that $q(t) \in \mathcal{M}_{>0}$ thus implies that $I \cap P_{=\mathbf{m}} = \emptyset$. Consequently, the two sets must be equal, $I \cap P_{<_{\mathcal{M}}\mathbf{m}} = I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$.

Finally, we prove the other direction for (b). Assume that $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| = \mathbf{m}$ and $I \cap P_{<_{\mathcal{M}}\mathbf{m}} = I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$. Since $|I \cap P_{<_{\mathcal{M}}\mathbf{m}}| = \mathbf{m}$, it is evident that $|I| \geq_{\mathcal{M}} \mathbf{m}$.

Suppose, for the sake of contradiction, that $|I| >_{\mathcal{M}} \mathbf{m}$. This implies that $|I| \not\leq_{\mathcal{M}} \mathbf{m}$ and so by (a) we conclude that $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| \not\leq_{\mathcal{M}} \mathbf{m}$. However, this contradicts the assumption that $I \cap P_{<_{\mathcal{M}}\mathbf{m}} = I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}$, which implies $|I \cap P_{\leq_{\mathcal{M}}\mathbf{m}}| = \mathbf{m}$. □

3.10 General Length and the Reflection Cocycle

Lemma 3.9.2 applies with $P = T$ under dominance order by Lemma 3.6.1, $\mathcal{M}$ the set of monomials with partial order as in section 3.1, and $q = p$. Also by Lemma 3.6.1, we can use $h' = h_{W,p,\mathscr{M}_\infty}$ and $P_{\leq_{\mathcal{M}}\mathbf{m}} = T_{\leq \mathbf{m}}$ defined in the previous section as they are compatible with our earlier notation for T.

Corollary 3.10.1. *Let* $\mathbf{m} \in \mathcal{M}$ *and* $x, y \in W$*.*

1. $l_{p,W}(x) \leq \mathbf{m}$ *iff* $|N_{\mathbf{m}}(x)| \leq \mathbf{m}$.

2. $l_{p,W}(x) = \mathbf{m}$ *iff* $|N_{\mathbf{m}}(x)| = \mathbf{m}$ *and* $\cup_{\mathbf{n} \lhd \mathbf{m}} N_{\mathbf{n}}(x) = N_{\mathbf{m}}(x)$.

3. $l_{p,W}(xy) \geq l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{m}$ *iff* $|N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)| \leq \mathbf{m}$.

4. $l_{p,W}(xy) < l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{m}$ *if* $|N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)| > \mathbf{m}$.

5. $l_{p,W}(xy) = l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{m}$ *iff* $|N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)| = \mathbf{m}$ *and* $\cup_{\mathbf{n} \lhd \mathbf{m}} (N_{\mathbf{n}}(x^{-1}) \cap N_{\mathbf{n}}(y)) = N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)$.

Proof. First, we note that if $w = r_1 ... r_n$ is a reduced expression, then $N(w) = \{t_i |\ i \in \{1, ..., n\}\}$ where $t_i := r_1 ... r_{i-1} r_i r_{i-1} ... r_1$ (see [19]). From this we see clearly that $p(r_i) = p(t_i)$ so that we have $l_{p,W} = |N(w)|$. Also, we remark that $N(x)$ is an ideal of T under dominance order by definition, and so for any finite family $(x_i)_{i \in I}$ of elements of W, the intersection $\cap_i N(x_i)$ is a finite ideal. From these remarks, (1) and (2) follow directly from the previous lemma.

The points (3) and (5) also follow using the cocycle condition $N(xy) = N(x) + xN(y)x^{-1}$ where + denotes symmetric difference. This condition along with the interpretation of $N(x)$ in terms of $l_{p,W}(x)$ imply that

$$l_{p,W}(xy) = l_{p,W}(x) + l_{p,W}(y) - 2|N(x^{-1}) \cap N(y)|.$$

Then let $I = N(x^{-1}) \cap N(y)$ and the Lemma implies (3) and (5).

Finally, if $|N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)| > \mathbf{m}$ then by (3) we know $l_{p,W}(xy) \ngeq l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{m}$. However, we also know that $|N(x^{-1}) \cap N(y)| = \mathbf{k} \geq |N_{\mathbf{m}}(x^{-1}) \cap N_{\mathbf{m}}(y)| > \mathbf{m}$ due to the containment of sets. So since

$$l_{p,W}(x) + l_{p,W}(y) + 2\mathbf{m} < l_{p,W}(x) + l_{p,W}(y) + 2\mathbf{k}$$

we have that

$$l_{p,W}(xy) = l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{k} < l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{m}$$

as required for (4). □

We finish this section with a lemma that has the same ideas as the previous lemma.

Lemma 3.10.2. *Suppose that $r, s \in S$ and $x, y \in W$ with $l_{p,W}(xr) > l_{p,W}(x)$ and $l_{p,W}(ys) > l_{p,W}(y)$. If $ysy^{-1} \preceq xrx^{-1}$, then $l_{p,W}(y^{-1}x) \leq l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{n}$ where $\mathbf{n} = h_{W,p,\mathscr{M}_\infty}(ysy^{-1})$.*

Proof. We have $s = y^{-1}ysy^{-1}y \preceq y^{-1}xrx^{-1}y$ since $l_{p,W}(ys) > l_{p,W}(y)$. Therefore, we know that $l_{p,W}(y^{-1}xr) = l_{p,W}(y^{-1}x) + l_p(r)$. Thus, by the exchange condition we know that $l_{p,W}(sy^{-1}xr) \geq l_{p,W}(y^{-1}x) + l_p(r) - l_p(s)$.

Now, $ysy^{-1} \preceq xrx^{-1}$ and $xrx^{-1} \in N(xr)$ so we must have $ysy^{-1} \in N(xr)$. Also, $ysy^{-1} \in N(ys)$ by definition. So

$$ysy^{-1} \in N(xr) \cap N(ys) = N(xr) \cap N((sy^{-1})^{-1}).$$

Therefore, we see that if $\mathbf{n} = h_{W,p,\mathscr{M}_\infty}(ysy^{-1})$, we have $|N((sy^{-1})^{-1}) \cap N(xr)| \geq \mathbf{n} + p(s)$ using Lemma 3.6.1. As in the proof of (4) from the previous lemma, this implies that $l_{p,W}(sy^{-1}xr) \leq l_{p,W}(sy^{-1}) + l_{p,W}(xr) - 2(\mathbf{n} + p(s))$. Putting together the information we have, we get

$$\begin{aligned} l_{p,W}(y^{-1}x) &\leq l_{p,W}(sy^{-1}xr) - l_p(r) + l_p(s) \\ &\leq l_{p,W}(xr) + l_{p,W}(sy^{-1}) - 2(\mathbf{n} + p(s)) - l_p(r) + l_p(s) \\ &\leq l_{p,W}(x) + l_p(r) + l_{p,W}(y) + l_p(s) - 2\mathbf{n} - 2p(s) - l_p(r) + l_p(s) \\ &= l_{p,W}(x) + l_{p,W}(y) - 2\mathbf{n}. \end{aligned}$$

□

CHAPTER 4

FINITE STATE AUTOMATA

In this chapter, we begin by introducing some standard terminology involving finite state automata. For the basic theory of using automata in groups, we use [21]. In [5], Brink and Howlett described a method of creating a finite state automata that would recognize any finitely generated Coxeter system. For any finite Coxeter system, the corresponding automata can be described by using the Cayley graph of the group.

More recently, Dyer ([18]) has demonstrated a much larger class of finite state automata that can recognize many different subsets of Coxeter systems as well. We generalize the results of Dyer using generalized length functions from Chapter 3. These length functions give rise to large class of finite state automata, which are more powerful than the family of automata introduced by Dyer. We then describe the corresponding languages (subsets of the free monoid S^*) recognized by our automata. These automata will be used in Chapter 5 to show that many subsets of Coxeter systems are regular.

4.1 Regular Languages

We use this section to introduce the idea of a regular language of a monoid. Let M be a monoid. By a *(left) M-set*, we mean a set A along with a function

$M \times A \to A$ given by $(m, a) \mapsto ma$ such that $1_M a = a$ for all $a \in A$ if 1_M is the identity of M and $m_1(m_2 a) = (m_1 m_2)a$ for all $a \in A$ and $m_i \in M$. In this case, we say that M acts on A.

A subset M' of M is called *(monoid) regular* if there is a left M-set A with A finite, an element $a_0 \in A$ and a subset $A_f \subset A$ with the property that $M' = \{m \in M | \ ma_0 \in A_f\}$. The following lemma is well known (see [21]). We include a proof here for completeness.

Lemma 4.1.1. *The family of regular subsets of M forms a Boolean subalgebra $\mathscr{R}(M)$ of $\mathscr{P}(M)$. If M is finitely (or countably) generated, then $\mathscr{R}(M)$ is countable.*

Proof. Suppose that M_1 and M_2 are both monoid regular subsets of M. We will demonstrate that $M_1^c = M \setminus M_1$, $M_1 \cup M_2$ and $M_1 \cap M_2$ are all monoid regular as well.

Since M_i is monoid regular, there exists M-sets A_1 and A_2 with $a_i \in A_i$ and $(A_i)_f \subset A_i$ such that $M_i = \{m \in M | \ ma_0 \in (A_i)_f\}$. Now, A_1 is finite and so $A'_f = A_1 \setminus (A_1)_f$ is also finite. Then $M_1^c = \{m \in M | \ ma_1 \in A'_f\}$ so M_1^c is regular. Next, let $B = A_1 \times A_2$ and let $m(a, a') = (ma, ma')$ be the action of M on B. Note that B is a finite set. Then $M_1 \cap M_2 = \{m \in M | \ ma_1 \in (A_1)_f; ma_2 \in (A_2)_f\} = \{m \in M | \ m(a_1, a_2) \in (A_1)_f \times (A_2)_f\}$. Let $B_f = (A_1)_f \times (A_2)_f\}$ and then $M_1 \cap M_2$ is regular. Finally, $M_1 \cup M_2 = \{m \in M | \ ma_1 \in (A_1)_f;$ or $ma_2 \in (A_2)_f\}$. Let $B'_f = (A_1)_f \times A_2 \cup A_1 \times (A_2)_f \subset B$ and then $M_1 \cup M_2 = \{m \in M | \ m(a_1, a_2) \in B'_f\}$ so $M_1 \cup M_2$ is regular. $\square$

Now, let S be a set, called an alphabet. Elements of the free monoid S^* on S are called words. We call subsets of S^* languages. Let $\mathscr{N}$ be an additively written commutative monoid and let map $L' : S \to \mathscr{N}$ be a map. Then, for any word $s_n...s_1 \in S^*$, we can define the length $\ell : S^* \to \mathscr{N}$ by $\ell(s_n...s_1) := L'(s_n) + \cdots + L'(s_1)$. For example, if we let $\mathscr{N} = \mathbb{N}$ and $L'(s) = 1$ for all $s \in S$, then $\ell(s_n...s_1) = n$. Also, for any word $w = s_n...s_1 \in S^*$ we define the transpose $\dagger : S^* \to S^*$ by $\dagger(w) := w^\dagger := s_1...s_n$.

Now, if A is a finite S^*-set, then we can describe the action by restriction of the multiplication map $S^* \times A \to A$ to a function $\mu : S \times A \to A$. Typically, μ is called the transition function of the S^*-set A. Since the transition function μ can be an arbitrary function, we get a bijection between functions $\mu : S \times A \to A$ where A is a set and S^*-sets A. If S is a finite set, then we define a *regular language* on S to be a monoid regular subset of S^*. Therefore, a regular language is determined by a tuple $(\mu : S \times A \to A, a_0, A_f)$ where A is a finite set, μ is a function, $a_0 \in A$ and $A_f \subseteq A$. The language determined by previous tuple is $\mathscr{L} = \{m \in S^* | \; ma_0 \in A_f\}$ where the S^* action is determined by μ.

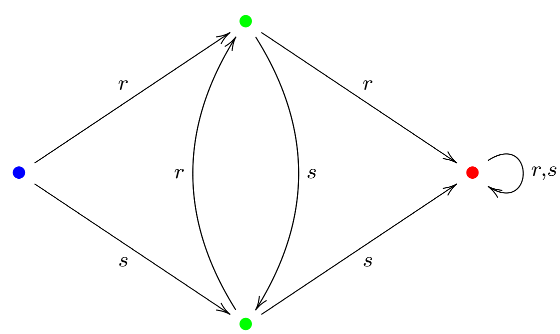
4.2 Automata

We can equivalently describe the monoid regular subsets of S^* by the terminology of finite state automata. A finite state automaton for the alphabet S is a tuple $(A, a_0, \mu : S \times A \to A, A_f)$ such that A is a finite set with a distinguished element a_0 called the initial state, a finite subset $A_f \subseteq A$ called the accepting states (or final states) and a transition function μ. The automaton reads a word $w = s_n...s_1$ from right to left, one letter at a time. The machine begins in the initial state a_0; upon reading a letter s_i, the state moves from the current state, a, to the state $\mu(s_i, a)$. We say the automaton accepts a word $w = s_n...s_1$ if after reading w, the current state of the automaton is a member of A_f. The set of words accepted by the automaton is called the language accepted by the automaton. It is clear that the language accepted by the automaton $(A, a_0, \mu : S \times A \to A, A_f)$ corresponds to the regular subset of S^* associated to A viewing A as an S^*-set.

Example 4.2.1. Consider the affine Weyl group of type $\widetilde{A}_1 = \langle r, s \mid r^2 = s^2 = 1 \rangle$. We know that the set of reduced expressions in $\widetilde{A}_1$ is given by

$$\{e, r, s, rs, sr, rsr, srs, rsrs, srsr, ...\}.$$

Below, we have a finite state automaton that recognizes the set of reduced expressions of words in $\widetilde{A}_1$. A is the vertex set of the given graph, a_0 is the blue vertex, and A_f is the set of non-red vertices. $\mu : S \times A \to A$ is given by the labeled edges.



4.3 Canonical Automata for Coxeter Systems

At this point, we want to describe some finite state automata for Coxeter systems.

Almost all objects defined in the rest of this chapter depend on the choice of p as in 3.2. We indicate when defining anything if it is dependent on p by including an appropriate subscript or superscript. However, when using the terminology, if there is no confusion about which function p we are referring to, we will leave off the corresponding subscript or superscript for ease of notation. Also, since (W, S) will be fixed, we let $l_p := l_{p,W}$. Recall, that l_p takes values in $\mathcal{M}$ (which depends on p) and that $\mathcal{M}$ has a partial order, which is described in section 3.1. In this chapter, we view $\mathcal{M}$ additively; therefore, $\mathcal{M} \cong \mathbb{N}^k$ and $\mathcal{M} \subset \mathcal{M}' \cong \mathbb{Z}^k$ (see section 3.1).

Let (W, S) be a Coxeter system and S^* be the free monoid on S. We denote the natural monoid homomorphism from S^* to W extending the identity map on S by $\nu : S^* \to W$. Let $\ell_p : S^* \to \mathcal{M}$ be the length of $w = s_1 \cdots s_n \in S^*$ given by $\ell_p(s_1 \cdots s_n) = p(s_1) + \cdots + p(s_n)$. Note $\ell_p(w) \geq l_{p,W}(\nu(w))$.

We associate to (W, S) an S^*-set $\mathscr{A} = \mathscr{A}_{(W,S)}$. As a set we have

$$\mathscr{A}_{(W,S,p)} := \mathscr{A} = \{\infty\} \cup \{(\mathbf{m}, A) |\ \mathbf{m} \in \mathcal{M},\ A \subseteq T_{\leq \mathbf{m}}\}.$$

The S^*-action can be described by its restriction to a function $S \times \mathscr{A} \to \mathscr{A}$ given by $(s, \infty) \mapsto \infty$ and

$$(s, (\mathbf{m}, A)) \mapsto \begin{cases} \infty, & \text{if } p(s) \not\leq \mathbf{m} \text{ and } s \in A \\ \left(\mathbf{m} - p(s), (sAs \setminus \{s\}) \cap T_{\leq(\mathbf{m}-p(s))}\right), & \text{if } p(s) \leq \mathbf{m} \text{ and } s \in A \\ \left(\mathbf{m}, (sAs \cup \{s\}) \cap T_{\leq \mathbf{m}}\right), & \text{if } s \notin A \end{cases}$$

We may define sub-S^*-sets $\mathscr{A}_{\mathbf{n}} = \mathscr{A}_{(W,S,p,\mathbf{n})}$ of $\mathscr{A}$ for $\mathbf{n} \in \mathcal{M}$ defined by

$$\mathscr{A}_{\mathbf{n}} := \{\infty\} \cup \{(\mathbf{m}, A) |\ \mathbf{m} \leq \mathbf{n},\ A \subseteq T_{\leq \mathbf{m}}\}$$

We know that if S is finite then $\mathscr{A}_{\mathbf{n}}$ is a finite S^*-set.

Lemma 4.3.1. *Let $x \in W$, $w \in S^*$, and $\mathbf{m} \in \mathcal{M}$. Then $w \cdot \infty = \infty$ and*

$$w \cdot (\mathbf{m}, N_{\mathbf{m}}(x)) \mapsto \begin{cases} \infty, & \text{if } \mathbf{k} \not\leq \mathbf{m} \\ (\mathbf{m} - \mathbf{k}, N_{\mathbf{m}-\mathbf{k}}(\nu(w)x)), & \text{if } \mathbf{k} \leq \mathbf{m} \end{cases}$$

in $\mathscr{A}$, where $\mathbf{k} := \frac{l_p(x)+\ell_p(w)-l_p(\nu(w)x)}{2}$

Proof. This follows by induction on $\ell_p(w)$ since it is clearly true for each $w = s \in S$. Then suppose that $w = sw'$ and the result is known for sw'. Then, by definition of the S-action on pairs, the result is clear. □

Finally, from the previous lemma, we can define

$$\mathfrak{M} = \mathfrak{M}_{(W,S,p)} := \{\infty\} \cup \{(\mathbf{m}, N_{\mathbf{m}}(x)) |\ \mathbf{m} \in \mathcal{M},\ x \in W\}$$

and $\mathfrak{M}_{(W,S,p,\mathbf{n})} = \mathfrak{M}_{\mathbf{n}} = \mathfrak{M} \cap \mathscr{A}_{\mathbf{n}}$ for $\mathbf{n} \in \mathcal{M}$, which are all sub-$S^*$-sets of $\mathscr{A}$. We call these the canonical and $\mathbf{n}$-canonical S^*-sets for (W, S) respectively.

4.4 Some Subsets of S^*

Let $A_{a,b} := \{w \in S^* |\ w \cdot a = b\}$ for any $a, b \in \mathfrak{M}$. By Lemma 4.3.1, we see that $A_{\infty,\infty} = S^*$ and $A_{\infty,x} = \emptyset$ for $x \neq \infty$. We now intend to describe the sets $A_{a,b}$. To do this, we introduce some subsets of S^* in the following way.

To begin with, for any word $w \in S^*$, we define

$$L_p(w) := \frac{\ell_p(w) - l_p(\nu(w))}{2} \in \mathcal{M}.$$

For any $\mathbf{m} \in \mathcal{M}$, we can then define $S^*_{\mathbf{m}} := S^*_{p,\mathbf{m}} = \{w \in S^* |\ L_p(w) = \mathbf{m}\}$. The elements of $S^*_{\mathbf{m}}$ will be called the $\mathbf{m}$-nonreduced words of (W, S). If S is finite, then for any $w \in W$, there are only finitely many $\mathbf{m}$-nonreduced expressions of w, namely the elements of the set $\nu^{-1}(w) \cap S^*_{\mathbf{m}}$.

Next, for $x, y \in W$ and $\mathbf{n} \in \mathcal{M}'$, we define more general sets

$$F^p_{x,y,\mathbf{n}} := \{w \in S^* |\ l_p(y\nu(w)x) = l_p(y) + \ell_p(w) + l_p(x) - 2\mathbf{n}\} \tag{4.4.1}$$

and also define, for $\mathbf{m} \in \mathcal{M}'$

$$F^p_{x,y,\mathbf{n},\mathbf{m}} := S^*_{\mathbf{m}} \cap F^p_{x,y,\mathbf{n}+\mathbf{m}}. \tag{4.4.2}$$

With this notation, we see that $S^*_{\mathbf{m}} = F_{1,1,\mathbf{m}}$.

Remark 4.4.1. It is clear from the definition that $F^p_{x,y,\mathbf{n}} = \emptyset$ if $\mathbf{n} \notin \mathcal{M}$, i.e. if $\mathbf{n} < \mathbf{0}$.

Example 4.4.2. For $s \in S$, we have that the set $F_{1,s,l_p(s)}$ is given by

$$\{w \in S^* | l_p(sw) = l_p(s) + \ell_p(w) - 2l_p(s) = \ell_p(w) - l_p(s)\},$$

which can also be described by

$$[\nu^{-1}(\{w \in W|\ l_p(sw) < l_p(w)\}) \cap S^*_{\mathbf{0}}] \dot\cup [\nu^{-1}(\{w \in W|\ l_p(sw) > l_p(w)\}) \cap S^*_{l_p(s)}].$$

Next for any $x \in W$, $\mathbf{k} \in \mathcal{M}'$ and finite subsets A, B of T, we let

$$H^p_{x,\mathbf{k},A,B} := \{w \in S^* |\ l_p(\nu(w)x) = \ell_p(w) + l_p(x) - 2\mathbf{k};\ N(\nu(w)x) \cap A = B\} \tag{4.4.3}$$

so it is clear the $H^p_{x,\mathbf{k},A,B} = \emptyset$ unless $\mathbf{k} \in \mathcal{M}$, $B \subseteq A$, and there exists a $y \in W$ such that $N(y) \cap A = B$.

According to Lemma 4.3.1, we see that for $\mathbf{m} \in \mathcal{M}$ and $x \in W$ we get

$$A_{(\mathbf{m},N_{\mathbf{m}}(x)),\infty} = \bigcup_{\mathbf{k} \not\leq \mathbf{m}} F_{x,1,\mathbf{k}}$$

and then for $\mathbf{m}, \mathbf{n} \in \mathcal{M}$,

$$A_{(\mathbf{m},N_{\mathbf{m}}(x)),(\mathbf{n},B)} = \begin{cases} \emptyset & \text{if } \mathbf{n} \not\leq \mathbf{m} \\ H_{x,\mathbf{m}-\mathbf{n},T_{\leq \mathbf{n}},B} & \text{if } \mathbf{n} \leq \mathbf{m} \end{cases} \tag{4.4.4}$$

where both sides are empty unless B is of the form $N_{\mathbf{n}}(y)$ for some $y \in W$.

4.5 The Boolean Algebra of Regular Sets of S^*

At this point, we assume that S is finite. Then, for $a, b \in \mathfrak{M}$, we can view $A_{a,b}$ in the following way. We choose $\mathbf{m}$ large enough so that $a, b \in \mathfrak{M}_{\mathbf{m}}$. Then $A_{a,b}$ is the regular language accepted by an automaton where $\mathfrak{M}_{\mathbf{m}}$ is the state set, the transition function described in 4.3.1, initial state a, and single accept state b. At this stage, we characterize the Boolean subalgebra of S^* generated by all $A_{a,b}$ where $a \neq \infty, b \neq \infty$.

Lemma 4.5.1. *The following Boolean subalgebras of S^* are all equal:*

1. *The Boolean subalgebra $\mathscr{D}_p$ of $\mathscr{P}(S^*)$ generated by all sets $A_{a,b}$ where $a, b \in \mathfrak{M} \setminus \{\infty\}$.*
2. *The Boolean subalgebra $\mathscr{D}'_p$ of $\mathscr{P}(S^*)$ generated by all sets $H_{x,\mathbf{k},A,B}$.*
3. *The Boolean subalgebra $\mathscr{D}''_p$ of $\mathscr{P}(S^*)$ generated by all sets $F_{x,y,\mathbf{k},\mathbf{m}}$.*
4. *The Boolean subalgebra $\mathscr{D}'''_p$ of $\mathscr{P}(S^*)$ generated by all sets $F_{x,y,\mathbf{k}}$.*

Moreover, $\mathscr{D}_p$ is stable under the automorphism $\dagger : S^* \to S^*$ given by $\dagger(s_n....s_1) = s_1...s_n$.

Proof. According to equation (4.4.4), we know that $\mathscr{D}_p$ is generated by all the sets $H_{x,\mathbf{k},T_{\leq \mathbf{n}},B}$ where $x, y \in W$, $\mathbf{n}, \mathbf{k} \in \mathcal{M}$ and $B \subset T$ is finite. Suppose that $\mathbf{k} \in \mathcal{M}$, $x \in W$, and $A, B \subseteq T$ are finite sets. We choose an element $\mathbf{n} \in \mathcal{M}$ such that $A \subseteq T_{\leq \mathbf{n}}$. Then

$$H_{x,\mathbf{k},A,B} = \bigcup_{\substack{B' \subseteq T_{\leq \mathbf{n}} \\ B' \cap A = B}} H_{x,\mathbf{k},T_{\leq \mathbf{n}},B'}$$

and so $H_{x,\mathbf{k},A,B}$ is generated by a finite union of $A_{a,b}$ for some $a, b \in \mathfrak{M} \setminus \{\infty\}$. Thus $\mathscr{D}_p = \mathscr{D}'_p$.

Next, we note that for $w \in F_{x,y,\mathbf{k},\mathbf{m}}$ we get (since $w \in S^*_{\mathbf{m}}$)

$$\begin{aligned} l_p(y\nu(w)x) &= l_p(y) + \ell_p(w) + l_p(x) - 2(\mathbf{k} + \mathbf{m}) \\ &= l_p(y) + l_p(\nu(w)) + 2\mathbf{m} + l_p(x) - 2(\mathbf{k} + \mathbf{m}) \end{aligned}$$

and so $l_p(y\nu(w)x) = l_p(y) + l_p(\nu(w)) + l_p(x) - 2\mathbf{k}$. This implies that $F_{x,y,\mathbf{k},\mathbf{m}} = \emptyset$ unless $\mathbf{0} \leq \mathbf{k} \leq l_p(y) + l_p(x)$. This shows that $S^*_{\mathbf{m}} = \cup_{\mathbf{0} \leq \mathbf{k} \leq l_p(x) + l_p(y)} F_{x,y,\mathbf{k},\mathbf{m}}$ and that we also have

$$F_{x,y,\mathbf{k}} = \bigcup_{\mathbf{k} - l_p(x) - l_p(y) \leq \mathbf{m} \leq \mathbf{k}} F_{x,y,\mathbf{k}-\mathbf{m},\mathbf{m}}$$

thus showing that $\mathscr{D}''_p = \mathscr{D}'''_p$ since it is clear that $F_{x,y,\mathbf{k},\mathbf{m}} = S^*_{\mathbf{m}} \cap F_{x,y,\mathbf{k}+\mathbf{m}} = F_{1,1,\mathbf{m}} \cap F_{x,y,\mathbf{k}+\mathbf{m}}$.

Now, we show that $\mathscr{D}'''_p \subseteq \mathscr{D}'_p$. For fixed $x \in W$ and $A \subseteq T$ finite, we have

$$S^* = \bigcup_{\mathbf{k} \in \mathcal{M}} \bigcup_{B \subseteq A} H_{x,\mathbf{k},A,B}$$

Recall from section 3.10 the terminology $|A| \in \mathcal{M}$ for $A \subseteq T$ finite. Let $y \in W$.

Then for $w \in H_{x,\mathbf{k},A,B}$ we have

$$l_p(y\nu(w)x) = l_p(y) + l_p(\nu(w)x) - 2|N(\nu(w)x) \cap N(y^{-1})|$$
$$= l_p(x) + \ell_p(w) + l_p(y) - 2\mathbf{k} - 2|N(\nu(w)x) \cap N(y^{-1})|.$$

From this we can see that

$$F_{x,y,\mathbf{m}} = \bigcup_{\mathbf{k},\mathbf{n}\in\mathcal{M}:\mathbf{k}+\mathbf{n}=\mathbf{m}} \ \bigcup_{B\subseteq N(y^{-1}):|B|=\mathbf{n}} H_{x,\mathbf{k},N(y^{-1}),B}.$$

This union is finite since there are only finitely many pairs $\mathbf{k}, \mathbf{n}$ in $\mathcal{M}$ less than $\mathbf{m}$ and $N(y^{-1})$ is a finite set, and so $\mathscr{D}_p''' \subseteq \mathscr{D}_p'$.

So we now must show that $\mathscr{D}'_p \subseteq \mathscr{D}'''_p$. Let $x, y \in W$ and $w \in S^*$, then we have

$$F_{x,1,\mathbf{k}} \cap F_{x,y,\mathbf{m}} = \{w \in W |\ l_p(\nu(w)x) = \ell_p(w) + l_p(x) - 2\mathbf{k} \text{ and}$$
$$l_p(y\nu(w)x) = l_p(y) + \ell_p(w) + l_p(x) - 2\mathbf{m}\}, \quad (4.5.1)$$

and by replacing $\ell_p(w) + l_p(x) = l_p(\nu(w)x) + 2\mathbf{k}$ in the second equation (using the first equation) we get

$$F_{x,1,\mathbf{k}} \cap F_{x,y,\mathbf{m}} = \{w \in W |\ l_p(\nu(w)x) = \ell_p(w) + l_p(x) - 2\mathbf{k} \text{ and}$$
$$l_p(y\nu(w)x) = l_p(\nu(w)x) + l_p(y) - 2\mathbf{m} + 2\mathbf{k}\}. \quad (4.5.2)$$

Now, we know that $l_p(\nu(w)x) \leq l_p(y) + l_p(y\nu(w)x) \leq l_p(y) + l_p(y) + l_p(\nu(w)x)$ consequently $\mathbf{0} \leq l_p(y) + l_p(y\nu(w)x) - l_p(\nu(w)x) \leq 2l_p(y)$. Substituting appropriate equations from (4.5.1) and (4.5.2), we note that $F_{x,1,\mathbf{k}} \cap F_{x,y,\mathbf{m}}$ is empty unless $\mathbf{0} \leq 2l_p(y) - 2\mathbf{m} + 2\mathbf{k} \leq 2l_p(y)$ and this implies that $\mathbf{k} \leq \mathbf{m} \leq \mathbf{k} + l_p(y)$. We also have $\cup_{\mathbf{m}\in\mathcal{M}} F_{x,y,\mathbf{m}} = S^*$. Now, let y from above be a given reflection $y = t$. Then we have that the following sets

$$L_{x,1,\mathbf{k},t} := \{w \in F_{x,1,\mathbf{k}} |\ t \in N(\nu(w)x)\} = F_{x,1,\mathbf{k}} \cap \bigcap_{\substack{\mathbf{m}:\ \mathbf{k} \le \mathbf{m} \le \mathbf{k}+l_p(t) \\ \mathbf{0} \le l_p(t) \le 2\mathbf{m}-2\mathbf{k}}} F_{x,t,\mathbf{m}}$$

and

$$L'_{x,1,\mathbf{k},t} := \{w \in F_{x,1,\mathbf{k}} |\ t \notin N(\nu(w)x)\} = F_{x,1,\mathbf{k}} \cap \bigcap_{\substack{\mathbf{m}:\ \mathbf{k} \le \mathbf{m} \le \mathbf{k}+l_p(t) \\ l_p(t) \ge 2\mathbf{m}-2\mathbf{k} \in \mathcal{M}}} F_{x,t,\mathbf{m}}$$

are clearly in $\mathscr{D}_p'''$ since these intersections are finite. If $B \subseteq A$, we thus get

$$H_{x,\mathbf{k},A,B} = \left(\bigcap_{t \in B} L_{x,1,\mathbf{k},t}\right) \cap \left(\bigcap_{t \in A \setminus B} L'_{x,1,\mathbf{k},t}\right),$$

and as we have noted before, $H_{x,\mathbf{k},A,B} = \emptyset$ if $B \not\subseteq A$. Therefore, $H_{x,\mathbf{k},A,B}$ is in $\mathscr{D}_p'''$, which is what we were trying to show.

Finally, suppose $w \in F_{x,y,\mathbf{m}}$ for some $\mathbf{m}$. Now, we have that $\nu(\dagger(w)) = \nu(w)^{-1}$. Since $l_p(y\nu(w)x) = l_p(y) + \ell_p(w) + l_p(x) - 2\mathbf{m}$ then we can reverse this to see that we must also have $l_p(x^{-1}\nu(\dagger(w))y^{-1}) = l_p(x^{-1}) + \ell_p(\dagger(w)) + l_p(y^{-1}) - 2\mathbf{m}$ and so $\dagger(w) \in F_{y^{-1},x^{-1},\mathbf{m}}$, which shows that $\mathscr{D}_p$ is stable under $\dagger$. □

4.6 The Reflection Cocycle and Automata

Recall that for $A \subseteq T$ a finite subset, we write $|A| = \sum_{t \in A} p(t)$. Thus, we note that $l_p(w) = |N(w)|$. Now, we state a lemma that describes the essence of the previous proof. The problem of determining, for all $w \in W$, the value of $l_p(xw) - l_p(w)$ for all x in some given finite subset of W is equivalent to finding, for all $w \in W$, the intersection $N(w) \cap A$ for some given finite subset A of reflections. Recall that $\mathcal{M}' = \mathbb{Z}^k$ if $\mathcal{M} = \mathbb{N}^k$.

Lemma 4.6.1. *(a) Let $x_1, ..., x_n$ be elements of W, and $\mathbf{l}_1, ..., \mathbf{l}_n \in \mathcal{M}'$. Let $A := \cup_{i=1}^n N(x_i^{-1})$. Then for $w \in W$, $l_p(x_i w) - l_p(w) = \mathbf{l}_i$ for all $i \in \{1, ..., n\}$ if and only if $|(N(w) \cap A) \cap N(x_i^{-1})| = \frac{l_p(x_i) - \mathbf{l}_i}{2}$ for all $i \in \{1, ..., n\}$.*

(b) Let A be a finite subset of T. Then for $w \in W$, we have

$$N(w) \cap A = \{t \in A |\ l_p(tw) = l_p(w) - \mathbf{k} \textit{ for some } \mathbf{k} \in \mathcal{M}_{\leq l_p(t)}\}$$

Proof. Recall that $N(xy) = N(x) + xN(y)x^{-1} = x(N(x^{-1}) + N(y))x^{-1}$. This implies that

$$l_p(xy) = l_p(x) + l_p(y) - 2|N(x^{-1}) \cap N(y)|$$

where $|-|$ is as above. Therefore, for $w \in W$, if $x_i \in W$, we have $l_p(x_i w) - l_p(w) = l_p(x) - 2|N(x_i^{-1}) \cap N(w)|$. Since $N(w) \cap N(x^{-1}) = (N(w) \cap A) \cap N(x_i^{-1})$, then $l_p(x_i w) - l_p(w) = \mathbf{l}_i$ if and only if $l_p(x_i) - 2|(N(w) \cap A) \cap N(x_i^{-1})| = \mathbf{l}_i$, and (a) follows.

Then, (b) follows since $N(w) = \{t \in T |\ l_p(tw) < l_p(w)\}$. We have $l_p(tw) = l_p(t) + l_p(w) - 2|N(t) \cap N(w)|$, we see that $l_p(t) - 2|N(t) \cap N(w)| = -\mathbf{k}$ for some $\mathbf{k} \in \mathcal{M}$. □

CHAPTER 5

REGULAR SUBSETS OF COXETER GROUPS

This chapter is devoted to the definition of various notions of regularity in Coxeter systems. We introduce three different types of regularity. The strongest type of regularity, p-complete regularity, will be examined in detail, and we will describe many subsets of Coxeter systems that are p-completely regular. The set of $\mathbf{m}$-nonreduced expressions of elements of any p-completely regular subset of W is a regular language in S^* for all $\mathbf{m} \in \mathcal{M}$. This implies that the multivariate Poincaré series of p-completely regular subsets are rational, providing a natural explanation (and strengthening) of known rationality results on Poincaré series of many natural subsets of W. We also use the results in this section to describe regularity of subsets in products of Coxeter systems.

5.1 Boolean Algebra of Regular Sets in W

For any $x, y \in W$ and $\mathbf{m} \in \mathcal{M}'$, we can now define a subset

$$W_{x,y,\mathbf{m}} := W^p_{x,y,\mathbf{m}} = \{w \in W |\ l_p(ywx) = l_p(y) + l_p(w) + l_p(x) - 2\mathbf{m}\} \subset W.$$

We have that $W = \bigcup_{\mathbf{0} \leq \mathbf{k} \leq l_p(x)+l_p(y)} W_{x,y,\mathbf{k}}$. Notice that $F_{x,y,\mathbf{k},\mathbf{m}} = \nu^{-1}(W_{x,y,\mathbf{k}}) \cap S^*_{\mathbf{m}}$ so that $\mathscr{D}_p$ is the Boolean subalgebra of $\mathscr{P}(S^*)$ generated by all intersections $\nu^{-1}(W_{x,y,\mathbf{k}}) \cap S^*_{\mathbf{m}}$.

Remark 5.1.1. Again, it is clear that $W_{x,y,\mathbf{m}} = \emptyset$ if $\mathbf{m} \notin \mathcal{M}$. This allows us to only consider sets $W_{x,y,\mathbf{m}}$ with $\mathbf{m} \geq \mathbf{0}$.

Let $\mathscr{B}_p(W)$ denote the Boolean subalgebra of $\mathscr{P}(W)$ generated by all the sets $W_{x,y,\mathbf{k}}$ with $x, y \in W$ and $\mathbf{k} \in \mathcal{M}$. Then $\mathscr{D}_p$ consists of all the sets $\cup_{\mathbf{m}\in\mathcal{M}}\nu^{-1}(B_{\mathbf{m}}) \cap S^*_{\mathbf{m}}$ where $B_{\mathbf{m}} \in \mathscr{B}_p(W)$ for all $\mathbf{m}$ and $B_{\mathbf{m}}$ is either $\emptyset$ for all but finitely many $\mathbf{m}$ or $B_{\mathbf{m}}$ is W for all but finitely many $\mathbf{m}$. Thus, we see that $\mathscr{D}_p$ is uniquely determined by the sets $\mathscr{B}_p(W)$ together with the set of $\mathbf{m}$-nonreduced expressions of w, for all $\mathbf{m} \in \mathcal{M}$ and $w \in W$.

Example 5.1.2. Let $J \subseteq S$ be a spherical subset, i.e. W_J is a finite standard parabolic subgroup. Then, let w_J be the longest element of W_J and suppose $l_p(w_J) = \mathbf{k}$. Then $D(J) := \{w \in W|\ D_L(w) = J\} \in \mathscr{B}_p(W)$ since it is the set $\{w \in W|\ l_p(w_J w) = l_p(w_J) + l_p(w) - 2\mathbf{k} = l_p(w) - l_p(w_J)\} = W_{1,w_J,\mathbf{k}}$. This set is known as the descent class associated to J.

5.2 Coxeter Groups are Automatic

The following result is due to Brink and Howlett, [5], in their proof that Coxeter groups are automatic.

Theorem 5.2.1. *Fix a total order of S and give S^* the corresponding reverse lexicographic order $\preceq_L$. Then for $w \in W$ define $\dot{w}$ as the minimal (under $\preceq_L$) element of the non-empty and finite set $\nu^{-1}(w) \cap S^*_{\mathbf{0}}$. Then $\dot{W} = \{\dot{w}|\ w \in W\}$ is a regular subset of S^*.*

5.3 Different Types of Regularity

Fix any map $\mu : W \to S^*$ with the following properties:

(i) For all $w \in W$, $\mu(w) \in S^*_{\mathbf{0}}$

(ii) $\nu \circ \mu = \mathrm{Id}_W$.

(iii) $\mu(W)$ is a regular subset of S^*.

Properties (i) and (ii) require that for $w \in W$, we have $\mu(w)$ is a reduced expression of W, and we know such a map exists due to Theorem 5.2.1.

We say that a subset A of W is μ-regular if $\mu(A)$ is a regular subset of S^*.

Definition 5.3.1. 1. A subset A of W is weakly regular if there is some map $w \mapsto \dot{w} : A \to S^*_{\mathbf{0}}$ such that $\nu(\dot{w}) = w$ for all $w \in A$ and $\{\dot{w} \mid w \in A\}$ is a regular subset of S^*.

2. A subset of W is regular (resp. p-strongly regular) if $\nu^{-1}(A) \cap S^*_{\mathbf{m}}$ is a regular subset of S^* for $\mathbf{m} = \mathbf{0}$ (resp. for all $\mathbf{m} \in \mathcal{M}$).

3. A subset of W is called p-completely regular if $A \in \mathscr{B}_p(W)$.

Example 5.3.2. Due to example 5.1.2, $D(J)$ is p-completely regular.

Lemma 5.3.3. *(a) Any p-strongly regular subset of W is regular.*

(b) If A is a regular subset of W, then A is μ-regular for any map μ satisfying conditions (i)-(iii) above.

(c) If A is μ-regular for some μ, then A is weakly regular.

Proof. Parts (a) and (c) follow directly from the definitions. To show part (b), we assume that A is a regular subset of W. This means that $\nu^{-1}(A) \cap S^*_{\mathbf{0}}$ is a regular subset of S^*. Additionally, since μ satisfies (i)-(iii), we know that $\mu(W)$ is a regular subset of S^* as well. Now, (b) follows since $\mu(A) = \mu(W) \cap (\nu^{-1}(A) \cap S^*_{\mathbf{0}})$ which is a finite intersection of regular sets and is thus regular. □

5.4 p-Regularity for Different p

Suppose that $p : T \to X_p$ and $p' : T \to X_{p'}$ are two different functions satisfying the properties of section 3.2 with corresponding sets of indeterminates X_p and $X_{p'}$ respectively. Additionally, let $\mathcal{M}_p$ and $\mathcal{M}_{p'}$ be the corresponding sets of monomials for X_p and $X_{p'}$. Since $\mathcal{M}_p$ and $\mathcal{M}_{p'}$ are the free commutative monoids on X_p and $X_{p'}$ respectively, then for any map $\varphi : X_p \to X_{p'}$ we get a corresponding map, denoted $\tilde{\varphi}$, with $\tilde{\varphi} : \mathcal{M}_p \to \mathcal{M}_{p'}$ given by the universal property. We note that $\deg(\mathbf{m}) = \deg(\tilde{\varphi}(\mathbf{m}))$ for any $\mathbf{m} \in \mathcal{M}_p$.

Proposition 5.4.1. *Let $\varphi : X_p \to X_{p'}$ be a surjective map such that $\varphi(p(t)) = p'(t)$ for all $t \in T$. Then the following are true:*

1. $\mathscr{D}_{p'} \subset \mathscr{D}_p$, *and*

2. $\mathscr{B}_{p'}(W) \subset \mathscr{B}_p(W)$.

Proof. For $q \in \{p, p'\}$, $\mathscr{D}_q$ is generated by the sets $F^q_{x,y,\mathbf{m}}$ with $x, y \in W$ and $\mathbf{m} \in \mathcal{M}_q$ according to Lemma 4.5.1. Now, since φ is surjective then $\tilde{\varphi}$ is surjective as well. Let $\mathbf{m} \in \mathcal{M}_{p'}$. Then define $Q := \tilde{\varphi}^{-1}(\mathbf{m})$. By surjectivity, $Q \neq \emptyset$ and since $\tilde{\varphi}$ preserves degree, $|Q| < \infty$. Then, it follows that

$$F^{p'}_{x,y,\mathbf{m}} = \bigcup_{\mathbf{k} \in Q} F^p_{x,y,\mathbf{k}} \in \mathscr{D}_p$$

and therefore $\mathscr{D}_{p'} \subset \mathscr{D}_p$, proving 1. Then, 2 is analogous replacing $F^q_{x,y,\mathbf{m}}$ by $W^q_{x,y,\mathbf{m}}$. □

5.5 Some Properties of p-Completely Regular Sets

In this section we describe some of the basic properties of p-completely regular sets.

Proposition 5.5.1. *(a) Any p-completely regular set is p-strongly regular (and thus also regular and μ-regular for any μ satisfying (i)-(iii) in section 5.3.*

(b) For any finite subsets $B, C \subseteq A$ of T the subset

$$G_{A,B,C} := \{w \in W |\ N(w) \cap A = B,\ N(w^{-1}) \cap A = C\}$$

of W is p-completely regular.

(c) If $A \in \mathscr{B}_p(W)$ then $A^{-1} := \{w^{-1} |\ w \in A\} \in \mathscr{B}_p(W)$.

(d) $\mathscr{B}_p(W)$ is closed under action by automorphisms of (W, S) that commute with p, i.e. automorphisms satisfying $\theta(p(t)) = p(\theta(t))$ for all $t \in T$.

(e) $\mathscr{B}_p(W)$ is closed under left and right multiplication by elements of W.

(f) $\mathscr{B}_p(W)$ contains all finite subsets of W.

(g) $\mathscr{B}_p(W)$ is countable.

Proof. Suppose that A is p-completely regular. Then as we have seen $\nu^{-1}(A) \cap S^*_{\mathbf{m}} \in \mathscr{D}_p$ for all $\mathbf{m} \in \mathcal{M}$. But $\mathscr{D}_p$ contains only regular languages by Lemma 4.5.1 part (1), which proves (a). To prove (b), we let $W_t := \{w \in W |\ l_p(tw) < l_p(w)\} = \bigcup_{\frac{l_p(t)-c(t)}{2} < \mathbf{k} \leq l_p(t)} W_{1,t,\mathbf{k}} \in \mathscr{B}_p(W)$ and $W'_t := \{w \in W |\ l_p(tw) > l_p(w)\} = \bigcup_{\mathbf{0} \leq \mathbf{k} < \frac{l_p(t)+c(t)}{2}} W_{1,t,\mathbf{k}} \in \mathscr{B}_p(W)$ since both are finite unions. We also have sets $X_t := \{w \in W |\ l_p(wt) < l_p(w)\}$ and $X'_t := \{w \in W |\ l_p(wt) > l_p(w)\}$, and both are in $\mathscr{B}_p(W)$ using finite unions of appropriate sets $W_{t,1,\mathbf{k}}$. Now, since A is finite, we get

$$Y_B := \bigcup_{t\in B} W_t \cap \bigcup_{t\in A\setminus B} W_t'$$

$$Y_C := \bigcup_{t\in C} X_t \cap \bigcup_{t\in A\setminus C} X_t'$$

and then $G_{A,B,C} := Y_B \cap Y_C$. In particular, this process describes an application, in detail, of Lemma 4.6.1.

Next, (c) holds since $(W_{x,y,\mathbf{k}})^{-1} = \{w^{-1} |\ l_p(ywx) = l_p(y)+l_p(w)+l_p(x)-2\mathbf{k}\} = \{w \in W |\ l_p(yw^{-1}x) = l_p(y) + l_p(w) + l_p(x) - 2\mathbf{k}\} = \{w \in W |\ l_p(x^{-1}wy^{-1}) = l_p(x^{-1}) + l_p(w) + l_p(y^{-1}) - 2\mathbf{k}\} = W_{y^{-1},x^{-1},\mathbf{k}}$.

Now, for any automorphism, θ, of (W, S) commuting with p, we have $l_p(w) = l_p(\theta(w))$ for all $w \in W$. Then, we see that if $w \in W_{x,y,\mathbf{k}}$ then $l_p(\theta(ywx)) = l_p(ywx) = l_p(y) + l_p(x) + l_p(w) - 2\mathbf{k} = l_p(\theta(y)) + l_p(\theta(x)) + l_p(\theta(w)) - 2\mathbf{k}$ so that $\theta(W_{x,y,\mathbf{k}}) \subset W_{\theta(x),\theta(y),\mathbf{k}}$. Using the same argument with the automorphism θ^{-1} demonstrates that in face $\theta(W_{x,y,\mathbf{k}}) = W_{\theta(x),\theta(y),\mathbf{k}}$, proving (d).

Next, (e) holds using the following observations. Let $a \in W$ and $b \in W$ and $x, y \in W$. Then $l_p(ya^{-1}) = l_p(y) + l_p(a^{-1}) - 2\mathbf{P}$ for some $\mathbf{P} \geq \mathbf{0}$ and $l_p(b^{-1}x) = l_p(x) + l_p(b^{-1}) - 2\mathbf{Q}$ for some $\mathbf{Q} \geq \mathbf{0}$. If $w \in W_{x,y,\mathbf{k}}$ we have $l_p(ywx) = l_p(y) + l_p(w) + l_p(x) - 2\mathbf{k}$. Then $awb \in W$ satisfies $l_p(ya^{-1}awbb^{-1}x) = l_p(y) + l_p(w)+l_p(x)-2\mathbf{k} = l_p(ya^{-1})-l_p(a^{-1})+2\mathbf{P}+l_p(w)-l_p(b^{-1})+l_p(b^{-1}x)+2\mathbf{Q}-2\mathbf{k} = l_p(ya^{-1}) + l_p(awb) - 2\mathbf{n} + l_p(b^{-1}x) + 2\mathbf{P} + 2\mathbf{Q} - 2\mathbf{k}$ where $\mathbf{0} \leq \mathbf{n} \leq l_p(a) + l_p(b)$ and $awb \in W_{b^{-1},a^{-1},\mathbf{n}}$. Therefore we have

$$aW_{x,y,\mathbf{k}}b = \bigcup_{\mathbf{0}\leq \mathbf{n} \leq l_p(a)+l_p(b)} (W_{b^{-1}x,ya^{-1},\mathbf{k}-\mathbf{P}-\mathbf{Q}+\mathbf{n}} \cap W_{b^{-1},a^{-1},\mathbf{n}})$$

where we recall that $W_{u,v,\mathbf{k}} = \emptyset$ if $\mathbf{k} < \mathbf{0}$.

Finally, (f) holds since we have $G_{S,\emptyset,\emptyset} = \{1\} \in \mathscr{B}_p(W)$ by (b). Then for any $w \in W$, $w\{1\} = \{w\} \in \mathscr{B}_p(W)$ by (e), and so any finite union of singleton elements of W is in $\mathscr{B}_p(W)$. Then, (g) follows from the nature of the fact that there are countably many sets $W_{x,y,\mathbf{k}}$ and we can only take finite unions and intersections. □

Example 5.5.2. For any $J \subseteq S$ spherical, the subgroup W_J is p-completely regular by (f). Since $W = W_{1,1,\mathbf{0}}$, then W is p-completely regular. If (W, S) is finite, we can easily demonstrate that (W, S) is regular by letting the Cayley graph of (W, S) be the corresponding S^*-set.

Remark 5.5.3. Given a completely regular set $A \subseteq W$, described explicitly as a finite union of finite intersections of the sets $W^p_{x,y,\mathbf{k}}$, we could construct an automaton on S accepting $\nu^{-1}(A) \cap S^*_{\mathbf{n}}$ for any $\mathbf{n} \in \mathcal{M}$ using Lemma 4.5.1. In addition, following from Lemma 5.3.3, we could construct an automaton on S accepting $\mu(A)$ for any μ satisfying the (i)-(iii) from 5.3.3.

5.6 Poincaré Series and Examples of Regular Subsets of W

The previous result leads to some results about the rationality of Poincaré series of subsets of Coxeter groups. First we recall the definition of a Poincaré series.

Definition 5.6.1. Let u be an indeterminate and consider the power series ring $\mathbb{Z}[[u]]$. For a subset $A \subseteq W$ we define the Poincaré series of A to be

$$P(A; W) = \sum_{w \in A} u^{\deg(l_p(w))}.$$

If we consider the multiplicative version of $\mathcal{M}$ generated by $\mathbf{X}$, then we can consider the completion $\mathbb{Z}[[\mathbf{X}]]$ of $\mathbb{Z}[\mathbf{X}]$. We define the multivariate Poincaré series of A to be

$$P_p(A;W) = \sum_{w \in A} l_p(w).$$

The following examples are previously known in the case where $l_p = l$ is the standard length; we use our results to show the examples are p-completely regular for general p as well.

Example 5.6.2. 1. For any finitely generated reflection subgroup W' and any standard parabolic subgroup W_J of (W,S), any double coset $W'wW_J$ has a unique element of minimal length determined by the conditions $N(w) \cap \chi(W') = \emptyset$ and $N(w^{-1}) \cap J = \emptyset$ where $\chi(W')$ is the set of canonical Coxeter generators of (W,S). In particular the set of these shortest coset representatives, denoted $W' \setminus W/W_J$ is thus given by the following:

$$W' \setminus W/W_J = \bigcup_{C \subset \chi(W)} G_{\chi(W'),\emptyset,C} \cap \bigcup_{B \subset J} G_{J,B,\emptyset}$$

and so is p-completely regular. This applies to shortest coset representatives $W' \setminus W$ of W and to shortest double coset representatives $W_K \setminus W/W_J$ of standard parabolic subgroups.

2. The weak right order $\leq$ on W is defined by $x \leq y$ if $N(x) \subseteq N(y)$ or equivalently if $l_p(x^{-1}y) = l_p(y) - l_p(x)$. Thus, if we fix $x \in W$ with $l_p(x) = \mathbf{m}$, the set

$$W_{1,x^{-1},\mathbf{m}} = \{y \in W |\ l_p(x^{-1}y) = l_p(y) + l_p(x) - 2l_p(x)\} = \{y \in W |\ y \geq x\}$$

is the upper rays in weak right order, and so this set is p-completely regular. A similar argument shoes that the upper rays in weak left order are also p-completely regular.

3. As we have seen in the proof of Proposition 5.5.1 the sets $\{w \in W |\ l_p(tw) < l_p(w)\}$ and $\{w \in W |\ l_p(wt) < l_p(w)\}$ are p-completely regular for fixed $t \in T$. These spaces (and their complements) are known as left and right half spaces respectively. It follows that any finite intersection and union of left (or right) half spaces is p-completely regular. See Figure 5.1 for example.

All of the sets in the previous examples thus have rational Poincaré series and multivariate Poincaré series due to a well-known result, called the transfer matrix method (see [2] or [28]), which states that regular languages have rational Poincaré series.

5.7 More Regular Subsets of W

The sets described in this section are shown to be p-completely regular in [18] for the case where $p : T \to \mathbb{N}$ with $p(t) = 1$ for all t. We now prove that these sets are also p-completely regular for general p.

Recall that W acts as a group of permutations on the set $T \times \{\pm 1\}$ such that for $s \in S$, $\epsilon \in \{\pm 1\}$, and $t \in T \setminus \{s\}$ we have $s(s,\epsilon) = (s, -\epsilon)$ and $s(t,\epsilon) = (sts, \epsilon)$. We note that as a $W \times \{\pm 1\}$-set, $T \times \{\pm 1\}$ is isomorphic to the standard root system Φ, with Φ^+ corresponding to $T \times \{1\}$ and simple roots Π corresponding to $S \times \{1\}$. With this terminology, we have the following p-completely regular sets.

Proposition 5.7.1. *1. For $u, v \in T$, $W(u,v) = \{w \in W |\ wuw^{-1} = v\}$ is p-completely regular.*

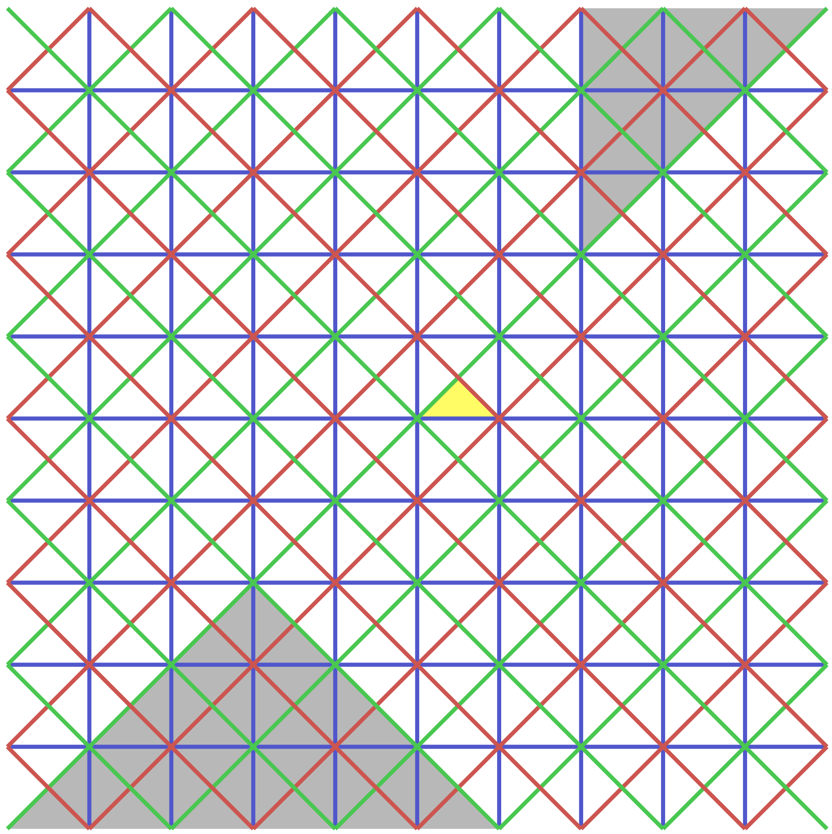


Figure 5.1. This diagram shows the tiling of the plane corresponding to $\widetilde{B}_2$ as in Figure 3.1. Each of the gray regions is an intersection of two half spaces, and thus each is p-completely regular for any p. Then, the total gray region is a union of two p-completely regular sets and so is p-completely regular itself. Therefore, this set also has a rational multivariate Poincaré series.

2. *For* $\alpha, \beta \in T \times \{\pm 1\}$, $W(\alpha, \beta) = \{w \in W |\ w(\alpha) = \beta\}$ *is* p*-completely regular.*

3. *For finite sets* $\Gamma, \Delta \subset T \times \{\pm 1\}$, *the set* $\{w \in W |\ w(\Gamma) = \Delta\}$ *is* p*-completely regular.*

4. *For* $J, K \subseteq S$, $\{w \in W |\ wJw^{-1} = K\}$ *is* p*-completely regular.*

5. *For* $J, K \subseteq S$, $\{w \in W |\ w((W_J \cap T) \times \{1\}) = (W_K \cap T) \times \{1\}\}$ *is* p*-completely regular.*

Proof. We first prove 1 for $u = r$ and $v = s$ with $r, s \in S$. In this case, we have $W(r, s) = \{w \in W |\ wrw^{-1} = s\} = \{w \in W |\ wr = sw\}$. Now, $W(r, s)$ will be empty (and thus p-completely regular) unless r is conjugate to s. So, if r and s are conjugate, then $l_p(r) = l_p(s)$ and according to the exchange condition, we have that $W(r, s) = Z_1 \cup Z_2$ where

$$Z_1 = \{w \in W |\ l_p(wr) = l_p(w) - l_p(r) = l_p(sw);\ l_p(swr) = l_p(w)\}$$

and

$$Z_2 = \{w \in W |\ l_p(wr) = l_p(w) + l_p(r) = l_p(sw);\ l_p(swr) = l_p(w)\}.$$

Then, each Z_i is p-completely regular due to the following descriptions.

$$Z_1 = W_{r,1,l_p(r)} \cap W_{1,s,l_p(s)} \cap W_{r,s,l_p(r)}$$

and

$$Z_2 = W_{r,1,\mathbf{0}} \cap W_{1,s,\mathbf{0}} \cap W_{r,s,l_p(r)}.$$

Now, in general, let $u = xrx^{-1}$ and let $v = ysy^{-1}$, then we know that $W(r,s)$ is p-completely regular by above and

$$\begin{aligned} yW(r,s)x^{-1} &= \{ywx^{-1} \in W|\ wrw^{-1} = s\} \\ &= \{w' \in W|\ y^{-1}w'xrx^{-1}w'^{-1}y = s\} \\ &= \{w' \in W|\ w'uw'^{-1} = v\} = W(u,v) \end{aligned}$$

is p-completely regular since p-completely regular sets are closed under left and right multiplication by elements of W by Proposition 5.5.1, proving 1.

Now, according to equation (2.2.3), we know that the W-action on $T \times \{\pm 1\}$ is given by $w(t,\epsilon) = (wtw^{-1}, \tau_{w,t}\epsilon)$ where $\tau_{w,t} \in \{\pm 1\}$ is given by $\tau_{w,t} = 1$ if $l_p(wt) > l_p(w)$ and $\tau_{w,t} = -1$ if $l_p(wt) < l_p(w)$. Then, let $\alpha = (u,\epsilon)$ and $\beta = (v,\epsilon')$ then

$$W(\alpha,\beta) = W(u,v) \cap \{w \in W|\ \tau_{w,u}\epsilon = \epsilon'\}.$$

Now, $W(u,v)$ is p-completely regular by 1. If $\epsilon = \epsilon'$ then $\tau_{w,u}$ must be 1 and so $\{w \in W|\ \tau_{w,u} = 1\} = \{w \in W|\ l_p(wu) > l_p(w)\}$ is p-completely regular by Example 3 in 5.6. Likewise if $\epsilon \neq \epsilon'$ then $\tau_{w,u}$ must be -1 and so $\{w \in W|\ \tau_{w,u} = -2\} = \{w \in W|\ l_p(wu) < l_p(w)\}$ is p-completely regular by the same example. Thus, $\{w \in W|\ \tau_{w,u}\epsilon = \epsilon'\}$ is p-completely regular so that $W(\alpha,\beta)$ also is as required.

Finally, to demonstrate 3, we note that the set described is empty unless $|\Gamma| = |\Delta|$. Let $\{\alpha_1, ..., \alpha_n\} = \Gamma$ and $\{\beta_1, ..., \beta_n\} = \Delta$. Then we have

$$Y_{\Gamma,\Delta} := \{w \in W|\ w(\Gamma) = \Delta\} = \cup_{\sigma \in S_n} \cap_{i=1}^{n} W(\alpha_i, \beta_{\sigma(i)}),$$

where S_n denotes the symmetric group. Then, $Y_{\Gamma,\Delta}$ is p-completely regular by 2.

Similarly, we have 4 following from 1 in the exact same way replacing $W(\alpha_i, \beta_{\sigma(i)})$ with $W(s_i, r_{\sigma(i)})$. Finally, according to section 2.3, we know that $w((W_J \cap T) \times \{1\}) = (W_K \cap T) \times \{1\}$ if and only if $w(J \times \{1\}) = K \times \{1\}$, so 5 follows from 3 directly using $\Gamma = J \times \{1\}$ and $\Delta = K \times \{1\}$ since both sets are finite. □

Example 5.7.2. The centralizer of a reflection, t, is $C_W(t) = \{w \in W |\ w^{-1}tw = t\}$. Due to the previous Proposition, this set is p-completely regular. So the centralizer of any finitely generated reflection subgroup is also p-completely regular.

5.8 Regularity in Direct Products

We now turn to looking at direct products of Coxeter systems. We have the following result describing the p-completely regular subsets of a Coxeter system in terms of the p-completely regular subsets of its irreducible components.

Lemma 5.8.1. *Suppose that (W, S) is the direct product of Coxeter systems (W_i, S_i) for $i = 1, ..., n$. We identify $W = W_1 \times \cdots \times W_n$ and $S = \dot{\cup} S_i$. Then $\mathscr{B}_p(W)$ is equal to the Boolean subalgebra $\mathscr{B}_p(W_1) * \cdots * \mathscr{B}_p(W_n)$ of $\mathscr{P}(W)$ generated by all subsets of W of the form $A_1 \times \cdots \times A_n$ with $A_i \in \mathscr{B}_p(W_i)$.*

Proof. This is clearly true for $n = 1$. Now, suppose it is true for $n = k$, we will show that it is true for $n = k + 1$. We consider $W' = W_1 \times \cdots \times W_k$ and W_{k+1} as subgroups of W. For any $x, y \in W$, we write $x = x'x_{k+1}$ and $y = y'y_{k+1}$ where

$x', y' \in W'$ and $x_{k+1}, y_{k+1} \in W_{k+1}$. Let $\mathbf{m} \in \mathcal{M}$. Then

$$\begin{aligned}
W^p_{x,y,\mathbf{m}} &= \{w \in W \mid l_p(ywx) = l_p(y) + l_p(w) + l_p(x) - 2\mathbf{m}\} \\
&= \{w'w_{k+1} \in W' \times W_{k+1} \mid \\
&\qquad l_p(y'w'x'y_{k+1}w_{k+1}x_{k+1}) = \\
&\qquad l_p(y') + l_p(y_{k+1}) + l_p(w') + l_p(w_{k+1}) + l_p(x') + l_p(x_{k+1}) - 2\mathbf{m}\} \\
&= \bigcup_{\substack{\mathbf{k}_1,\mathbf{k}_2 \in \mathcal{M} \\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{m}}} W'^p_{x',y',\mathbf{k}_1} \times (W_{k+1})^p_{x_{k+1},y_{k+1},\mathbf{k}_2}
\end{aligned} \tag{5.8.1}$$

so that $\mathscr{B}_p(W) \subseteq \mathscr{B}_p(W') * \mathscr{B}_p(W_{k+1})$. Also, equation (5.8.1) implies that $W^p_{x',y',\mathbf{k}_1} = W'^p_{x',y',\mathbf{k}_1} \times W_{k+1}$ and $W^p_{x_{k+1},y_{k+1},\mathbf{k}_2} = W' \times (W_{k+1})^p_{x_{k+1},y_{k+1},\mathbf{k}_2}$. Therefore, we have

$$W'^p_{x',y',\mathbf{k}_1} \times (W_{k+1})^p_{x_{k+1},y_{k+1},\mathbf{k}_2} = W^p_{x',y',\mathbf{k}_1} \cap W^p_{x_{k+1},y_{k+1},\mathbf{k}_2}$$

so that $\mathscr{B}_p(W') * \mathscr{B}_p(W_{k+1}) \subseteq \mathscr{B}_p(W)$. However, by induction, we know that $\mathscr{B}_p(W') = \mathscr{B}_p(W_1) * \cdots * \mathscr{B}_p(W_k)$ and the result follows. □

5.9 Product Regularity

Complete regularity of a subset in a product $W = W_1 \times \cdots \times W_n$ implies regularity of many subsets of S^* and so regularity of various subsets of $S_1^* \times \cdots \times S_n^*$. Then we regard each S_i^* as a submonoid of S^* and each W_i as a subgroup of W; then we can define natural homomorphisms $\nu_i : S_i^* \to W_i$. We say that a subset of $S_1^* \times \cdots \times S_n^*$ is product regular if it is in the Boolean algebra of subsets generated by the products $A_1 \times \cdots \times A_n$ with each A_i regular in S_i^*.

Lemma 5.9.1. *For any p-completely regular subset A of $W_1 \times \cdots \times W_n$, and any* *n-tuple* $(\mathbf{k}_1, ..., \mathbf{k}_n) \in \mathcal{M}^n$*, the subset*

$$\{(x_1, ..., x_n) \in (S_1)^*_{\mathbf{k}_1} \times \cdots \times (S_n)^*_{\mathbf{k}_n} |\ (\nu_1(x_1), ..., \nu_n(x_n)) \in A\}$$

is product regular in $(S_1)^* \times \cdots \times (S_n)^*$*.*

Proof. For any subset A of W, we define

$$A' := \{x_1, ..., x_n) \in (S_1)^*_{\mathbf{k}_1} \times \cdots \times (S_n)^*_{\mathbf{k}_n} |\ (\nu_1(x_1), ..., \nu_n(x_n)) \in A\}.$$

If A is a product then $A = A_1 \times \cdots \times A_n$, then we have

$$A' = \left((S_1)^*_{\mathbf{k}_1} \cap \nu_1^{-1}(A_1)\right) \times \cdots \times \left((S_n)^*_{\mathbf{k}_n} \cap \nu_n^{-1}(A_n)\right).$$

In this, if each A_i is p-completely regular (or just p-strongly regular), then each $(S_i)^*_{\mathbf{k}_i} \cap \nu_i^{-1}(A_i)$ is regular in $(S_i)^*$ by Lemma 5.3.3 and thus A' is product regular. Now if $A = \cup_{\mathbf{m}} B_m$, where the union is finite, then $A' = \cup_{\mathbf{m}} B'_m$. Now, this implies that A' is product regular for any p-completely regular set since any such set A is a finite union of products of p-completely regular sets. □

5.10 Multivariate Product Regularity

For any Coxeter system (W, S), define a subset of W^n to be p-completely regular if it is completely regular as a subset of the n-fold product Coxeter system $(W, S)^n$. We have the following result.

Lemma 5.10.1. *Let* n, m *be natural numbers,* $\mathbf{m} \in \mathcal{M}$*,* $f_1, ..., f_m \in \{1, ..., n\}$*,* $\epsilon_1, ..., \epsilon_m \in \{\pm 1\}$*,* $x_0, x_1, ..., x_m \in W$ *and* A, B, C *be finite subsets of* W *satisfying* $B, C \subseteq A$*. Then let* L *be the subset of* W^n *consisting of* n*-tuples satisfying the* *following conditions:*

(i) If

$$v := x_m(w_{f_m})^{\epsilon_m} x_{m-1}(w_{f_{m-1}})^{\epsilon_{m-1}} x_{m-2} \cdots x_1(w_{f_1})^{\epsilon_1} x_0$$

and $\mathbf{k} := \sum_{i=0}^{m} l_p(x_i) + \sum_{i=1}^{m} l_p(w_{f_i})$, *then* $l_p(v) = \mathbf{k} - 2\mathbf{m}$.

(ii) $N(v) \cap A = B$ *and* $N(v^{-1}) \cap A = C$.

Then L *is a* p*-completely regular subset of* W^n.

Proof. We will prove this by induction on m. To make things more tractable, we denote L by

$$L = L(n, m, \mathbf{m}, (f_j), (\epsilon_j), (x_j), A, B, C)$$

to show that L depends on its parameter set.

First, suppose that m is fixed. If

$$L(n, m, \mathbf{m}, (f_j), (\epsilon_j), (x_j), \emptyset, \emptyset, \emptyset)$$

is p-completely regular, then

$$L(n, m, \mathbf{m}, (f_j), (\epsilon_j), (x_j), A, B, C)$$

is also p-completely regular using Lemma 4.6.1 (similar to the proof of Proposition 5.5.1 part (b)).

Now, suppose that $m = 1$. The set

$$L(1, 1, \mathbf{m}, (1), (1), (x_0, x_1), \emptyset, \emptyset, \emptyset) = W^p_{x_0, x_1, \mathbf{m}}$$

and thus is p-completely regular by definition. Also, as in the proof of Proposition

5.5.1 part (c), we have

$$L(1,1,\mathbf{m},(1),(-1),(x_0,x_1),\emptyset,\emptyset,\emptyset) = L(1,1,\mathbf{m},(1),(1),(x_1^{-1},x_0^{-1}),\emptyset,\emptyset,\emptyset).$$

It follows that the sets

$$L(1,1,\mathbf{m},(1),(\epsilon),(x_0,x_1),\emptyset,\emptyset,\emptyset)$$

for $\epsilon \in \{\pm 1\}$ are p-completely regular.

Let $n \in \mathbb{N}$. Then any set

$$L' = L(n,1,\mathbf{m},(k),(\epsilon),(x_0,x_1),\emptyset,\emptyset,\emptyset)$$

with $k \in \{1,...,n\}$ is of the form

$$L' = W^{k-1} \times L(1,1,\mathbf{m},(1),(\epsilon),(x_0,x_1),\emptyset,\emptyset,\emptyset) \times W^{n-k}$$

and thus is p-completely regular. Therefore, we have shown that any set

$$L(n,1,\mathbf{m},(k),(\epsilon),(x_0,x_1),A,B,C)$$

is p-completely regular.

At this point, we have proved that the set L is p-completely regular if $m = 1$, and if $m = 0$ it is clear that L is p-completely regular. So, we now move to the induction step. We will show that for fixed m, p-complete regularity of all sets of the form

$$L(n,m,\mathbf{m},(f_j),(\epsilon_j),(x_j),A,B,C)$$

implies p-complete regularity of any sets of the form

$$L'' := L(n, m+1, \mathbf{m}, (f_j), (\epsilon_j), (x_j), \emptyset, \emptyset, \emptyset).$$

Now, suppose that $w = (w_1, ..., w_n) \in W^n$ is an element of L'' above. Then we define the following two words:

$$v_m := x_m (w_{f_m})^{\epsilon_m} x_{m-1} (w_{f_{m-1}})^{\epsilon_{m-1}} x_{m-2} \cdots x_1 (w_{f_1})^{\epsilon_1} x_0$$

and

$$v_{m+1} := x_{m+1} (w_{f_{m+1}})^{\epsilon_{m+1}} x_m (w_{f_m})^{\epsilon_m} x_{m-1} (w_{f_{m-1}})^{\epsilon_{m-1}} x_{m-2} \cdots x_1 (w_{f_1})^{\epsilon_1} x_0$$

and two p-lengths $\mathbf{k}_m := \sum_{i=0}^m l_p(x_i) + \sum_{i=1}^m l_p(w_{f_i})$ and $\mathbf{k}_{m+1} := \mathbf{k}_m + l_p(x_{m+1}) + l_p(w_{f_{m+1}})$. Let $v := x_{m+1}(w_{f_{m+1}})^{\epsilon_{m+1}}$ so that $v_{m+1} = v v_m$, and let $\mathbf{k} := l_p(x_{m+1}) + l_p(w_{f_{m+1}})$ so that $\mathbf{k}_{m+1} = \mathbf{k}_m + \mathbf{k}$. Using these equalities, we have

$$\begin{aligned} l_p(v_{m+1}) &= l_p(v) + l_p(v_m) - 2|N(v^{-1}) \cap N(v_m)| \\ &= \mathbf{k}_{m+1} - \mathbf{k} - \mathbf{k}_m + l_p(v) + l_p(v_m) - 2|N(v^{-1}) \cap N(v_m)| \\ &= \mathbf{k}_{m+1} - 2\left(\frac{\mathbf{k} - l_p(v)}{2} + \frac{\mathbf{k}_m - l_p(v_m)}{2} + |N(v^{-1}) \cap N(v_m)|\right). \end{aligned}$$

By definition, $l_p(v) = l_p(x_{m+1}) + l_p(w_{f_{m+1}}) - 2|N(x_{m+1}^{-1}) \cap N((w_{f_{m+1}})^{\epsilon_{m+1}})| \leq \mathbf{k}$, and similarly we have $l_p(v_m) \leq \mathbf{k}_m$. We know that $w \in L''$ so we must have $l_p(v_{m+1}) = \mathbf{k}_{m+1} - 2\mathbf{m}$.

According to Corollary 3.10.1 part (4) we know that $l_p(v_{m+1}) < \mathbf{k}_{m+1} - 2\mathbf{m}$ if $l_p(v) < \mathbf{k} - 2\mathbf{m}$, $l_p(v_m) < \mathbf{k}_m - 2\mathbf{m}$ or $|N_{\mathbf{m}}(v^{-1}) \cap N_{\mathbf{m}}(v_m)| > \mathbf{m}$. Otherwise, we have

$$l_p(v_{m+1}) = \mathbf{k}_{m+1} - 2\left(\frac{\mathbf{k} - l_p(v)}{2} + \frac{\mathbf{k}_m - l_p(v_m)}{2} + |N_{\mathbf{m}}(v^{-1}) \cap N_{\mathbf{m}}(v_m)|\right). \tag{5.10.1}$$

Suppose $l_p(v) = \mathbf{k} - 2\mathbf{m}'$ and $l_p(v_m) = \mathbf{k}_m - 2\mathbf{m}''$. Then the equation (5.10.1) is

$$l_p(v_{m+1}) = \mathbf{k}_{m+1} - 2\left(\mathbf{m}' + \mathbf{m}'' + |N_{\mathbf{m}}(v^{-1}) \cap N_{\mathbf{m}}(v_m)|\right).$$

Since $l_p(v_{m+1}) = \mathbf{k}_{m+1} - 2\mathbf{m}$, then we have the following:

$$L'' = \bigcup_{\substack{\mathbf{m}',\mathbf{m}'',B,C \\ B,C \subset T_{\leq \mathbf{m}} \\ \mathbf{m}'+\mathbf{m}''+|B\cap C|=\mathbf{m}}} L_{\mathbf{m}',\mathbf{m}'',B,C}$$

where

$$L_{\mathbf{m}',\mathbf{m}'',B,C} = L(n, m, \mathbf{m}', (f_j)_{j=1}^m, (\epsilon_j)_{j=1}^m, (x_j)_{j=0}^m, T_{\leq \mathbf{m}}, B, \emptyset)$$
$$\cap\, L(n, 1, \mathbf{m}'', (f_{m+1}), (\epsilon_{m+1}), (1, x_{m+1}), T_{\leq \mathbf{m}}, \emptyset, C)$$

both of which are p-completely regular by induction. Thus L'' is a finite union of p-completely regular sets and so L'' is p-completely regular as needed. □

CHAPTER 6

REGULARITY IN T AND MIXED PRODUCTS

The p-complete regularity in W as we have described has been useful for showing that many natural subsets of W are regular, and therefore have rational Poincaré series. However, there are many natural subsets of W, in particular subsets of T, that can be shown to not be p-complete regular in W (or even μ-regular in W). Nonetheless, we would like to be able to show that certain subsets of T are regular in some sense. We will use the fact that every reflection has a palindromic reduced expression to introduce a notion of p-complete regularity in T. We can then extend our notion of regularity to regularity in products of T or even mixed products of W and T. We include some examples of p-completely regular subsets of T, and the regularity implies that these subsets have rational (multivariate) Poincaré series.

6.1 Regularity of sets of Reflections

Let $X \subset S^*$ be the set of all non-empty words in S^*. We see that X is clearly a regular subset of S^* being the complement of the set consisting of the empty word. We say that a subset of X is regular if it is regular as a subset of S^*. Define $\eta : X \to T$ by $\eta(r_{n+1}r_n \cdots r_1) = r_1 \cdots r_n r_{n+1} r_n \cdots r_1$. Let $\ell'_p(x) = 2\ell_p(x) - l_p(r_{n+1}) = 2\ell_p(x) - p(\eta(x))$. Let $\preceq'_L$ be the restriction of some reverse lexicographic order on S^* to X.

Proposition 6.1.1. *1. For any* $\mathbf{m} \in \mathcal{M}$ *let* $X_{\mathbf{m}} := \{x \in X | \; l_p(\eta(x)) = \ell'_p(x) - 2\mathbf{m}\}$. *Then* $X_{\mathbf{m}}$ *is a regular subset of* X.

2. *For* $t \in T$, *define* $\dot{x}_t$ *as the minimal (under the order* $\preceq'_L$*) element of the non-empty finite set* $\eta^{-1}(t) \cap X_{\mathbf{0}}$. *Then* $\dot{T} = \{\dot{x}_t | \; t \in T\}$ *is a regular subset of* X.

Proof. We prove 1 first. Let $\mathbf{m} \in \mathcal{M}$, and let $x \in \mathbf{X_m}$. Then, we can write $x = rx'$ with $r \in S$ and $x' \in S^*_{\mathbf{k}}$ for some $\mathbf{k} \in \mathcal{M}$. By definition, we know that $l_p(\nu(x')) = \ell_p(x') - 2\mathbf{k}$. Since $x \in \mathbf{X_m}$, and substituting the previous formula we get

$$\begin{aligned} l_p(\nu(x')^{-1} r \nu(x')) = l_p(\eta(x)) &= \ell'_p(x) - 2\mathbf{m} = 2\ell_p(x') + l_p(r) - 2\mathbf{m} \\ &= 2(2l_p(\nu(x')) + 2\mathbf{k}) + l_p(r) - 2\mathbf{m} \qquad . \\ &= 2l_p(\nu(x')) + l_p(r) - 2(\mathbf{m} - 2\mathbf{k}) \end{aligned} \tag{6.1.1}$$

Equation (6.1.1) implies that $\mathbf{m} - 2\mathbf{k} \geq \mathbf{0}$ so $2\mathbf{k} \leq \mathbf{m}$; furthermore, (6.1.1) also implies that $\nu(x') \in A_{r,\mathbf{m}-2\mathbf{k}}$ where

$$A_{r,\mathbf{m}-2\mathbf{k}} := L(1, 2, \mathbf{m} - 2\mathbf{k}, (1, 1), (1, -1), (e, r, e), \emptyset, \emptyset, \emptyset)$$

where $e \in W$ is the identity, and L is as in 5.10. According to Lemma 5.10.1 $A_{r,\mathbf{m}-2\mathbf{k}}$ is p-completely regular. Therefore, we have $x' \in S^*_{\mathbf{k}} \cap \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}})$ which is regular by Lemma 5.5.1. So, for any $x \in X_{\mathbf{m}}$, there is an $r \in S$ and $\mathbf{k} \in \mathcal{M}$ with $2\mathbf{k} \leq \mathbf{m}$ such that $x \in r(S^*_{\mathbf{k}} \cap \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}}))$. Thus

$$X_{\mathbf{m}} \subseteq \bigcup_{r \in S} r \left(\bigcup_{0 \leq 2\mathbf{k} \leq \mathbf{m}} S^*_{\mathbf{k}} \cap \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}}) \right)$$

Now, suppose that $x \in r(S^*_{\mathbf{k}} \cap \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}}))$ for some $r \in S$ and $\mathbf{k} \in \mathcal{M}$ with $2\mathbf{k} \leq \mathbf{m}$. We have from above that

$$A_{r,\mathbf{m}-2\mathbf{k}} = \{w \in W | \; l_p(w^{-1}rw) = 2l_p(w) + l_p(r) - 2(\mathbf{m} - 2\mathbf{k})\}.$$

Thus $x = rx'$ with $x' \in \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}})$ and $l_p(\nu(x')) = 2\ell_p(x') - 2\mathbf{k}$. This gives that

$$\begin{aligned} l_p(\eta(x)) = l_p(\nu(x')^{-1} r \nu(x')) &= 2l_p(\nu(x')) + l_p(r) - 2(\mathbf{m} - 2\mathbf{k}) \\ &= 2(\ell_p(x') - 2\mathbf{k}) + l_p(r) - 2(\mathbf{m} - 2\mathbf{k}) \\ &= 2\ell_p(x') + l_p(r) - 2\mathbf{m} = 2\ell'_p(x) - 2\mathbf{m}, \end{aligned}$$

so that $x \in X_{\mathbf{m}}$. Thus we have shown

$$X_{\mathbf{m}} = \bigcup_{r \in S} r \left(\bigcup_{0 \leq 2\mathbf{k} \leq \mathbf{m}} S^*_{\mathbf{k}} \cap \nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}}) \right),$$

and the right hand side is a finite union of regular sets as already shown hence $X_{\mathbf{m}}$ is regular as well.

Part (2) follows from and unpublished result due to Bob Howlett, which is the analog of 5.2.1 for T instead of W. The result is mentioned in [18]. □

6.2 Types of Regularity in T

Fix any map $\tau : T \to X$, denoted $t \to x_t$ for $t \in T$, with the following properties:

(i) For all $t \in T$, $\tau(t) \in X_{\mathbf{0}}$.

(ii) $\eta \circ \tau = \mathrm{Id}_T$.

(iii) $\dot{T} := \tau(T)$ is a regular subset of X.

We know that maps, τ, satisfying (i)-(iii) above exist due to the previous proposition. We say a subset A of T is τ-regular if $\tau(A)$ is regular in X.

Definition 6.2.1. A subset A of T is regular (respectively p-strongly regular) in T if $\eta^{-1}(A) \cap X_{\mathbf{m}}$ is a regular subset of X for $\mathbf{m} = \mathbf{0}$ (respectively all $\mathbf{m} \in \mathcal{M}$).

Lemma 6.2.2. *(a) Any p-strongly regular subset of T is regular.*

(b) If A is a regular subset of T then A is τ-regular for any map satisfying (i)-(iii) above.

Proof. Part (a) is a restatement of the definition since $\mathbf{0} \in \mathcal{M}$. For part (b), we have that for regular subset $A \subseteq T$ and τ satisfying (i)-(iii) above, we get that $\tau(A) = (\eta^{-1}(A) \cap X_{\mathbf{0}}) \cap \tau(T)$ which is an intersection of regular sets and thus regular. □

6.3 p-Complete Regularity in T

We now want to describe the analog of $\mathscr{B}_p(W)$ when dealing with regular subsets of T. To do this, we first introduce a few subsets of $\mathscr{P}(T)$.

Definition 6.3.1. 1. For $x \in W$ and $\mathbf{m} \in \mathcal{M}$, we let

$$R^p_{x,\mathbf{m}} := \{t \in T |\ l_p(xtx^{-1}) = 2l_p(x) + l_p(t) - 2\mathbf{m}\}.$$

Then, let $\mathscr{B}''_p(T)$ be the Boolean subalgebra of $\mathscr{P}(T)$ generated by all sets $R^p_{x,\mathbf{m}}$ for $x \in W$ and $\mathbf{m} \in \mathcal{M}$.

2. For $t \in T$ and $A \subseteq T$, we define $A_t := \{w \in W |\ w^{-1}tw \in A\}$. Then, let $\mathscr{B}'_p(T)$ be the family of all subsets A of T such that for all $r \in S$, $A_r \in \mathscr{B}_p(W)$.

3. For any $t \in T$ and $A \subset T$ and $\mathbf{m} \in \mathcal{M}$, we define

$$A^p_{t,\mathbf{m}} := \{w \in W|\ w^{-1}tw \in A;\ l_p(w^{-1}tw) = 2l_p(w) + l_p(t) - 2\mathbf{m}\}.$$

We let $\mathscr{B}_p(T)$ be the family of all subsets A of T such that for all $r \in S$ and $\mathbf{m} \in \mathcal{M}$ we have $A_{r,\mathbf{m}} \in \mathscr{B}_p(W)$. We call the elements of $\mathscr{B}_p(T)$ the p-completely regular subsets of T.

Properties of these families of subsets are described in the following proposition. Again, for ease of notation, we omit the superscript p since p is understood to be fixed.

Proposition 6.3.2. *(a) If $A \in \mathscr{B}'_p(T)$ then $A_t \in \mathscr{B}_p(W)$ for all $t \in T$. Likewise, if $A \in \mathscr{B}_p(T)$ then $A_{t,\mathbf{m}} \in \mathscr{B}_p(W)$ for all $t \in T$ and $\mathbf{m} \in \mathcal{M}$.*

(b) $\mathscr{B}''_p(T), \mathscr{B}'_p(T) \subset \mathscr{B}_p(T)$ are Boolean subalgebras of $\mathscr{P}(T)$ closed under conjugation by elements of W and closed under permutations of T induced by automorphisms, θ, of (W, S) that commute with p.

(c) $\mathscr{B}'_p(T)$ contains all finite subsets of T and all conjugacy classes of simple reflections.

(d) Any set A which is p-completely regular in T is p-strongly regular in T.

Proof. Suppose $A \in \mathscr{B}'_p(T)$. Then $A_r \in \mathscr{B}_p(W)$ for all $r \in S$. Now, let $t \in T$ be given. Then, $t = u^{-1}su$ for some $u \in W$ and $s \in S$. Now we have

$$\begin{aligned} A_t &= \{w \in W|\ w^{-1}tw \in A\} = \{w \in W|\ w^{-1}u^{-1}suw \in A\} \\ &= \{u^{-1}v \in W|\ v^{-1}sv \in A\} = u^{-1} \cdot \{v \in W|\ v^{-1}sv \in A\} = u^{-1}A_s, \end{aligned}$$

and $u^{-1}A_s \in \mathscr{B}_p(W)$ since $A_s \in \mathscr{B}_p(W)$ and by Proposition 5.5.1 $\mathscr{B}_p(W)$ is closed under left multiplication by elements of W, thus the first part of (a) is true. For the second part of (a), we fix $u \in W$ and $s \in S$ such that $t = u^{-1}su$ and $l_p(t) = 2l_p(u) + l_p(s)$. Now, we consider the sets

$$B_{\mathbf{k}} := u^{-1}A_{s,\mathbf{m}-2\mathbf{k}} \cap W_{1,u,k} \in \mathscr{B}_p(W).$$

Notice, if $w \in B_{\mathbf{k}}$ then $w = u^{-1}v$ with $v \in A_{s,\mathbf{m}-2\mathbf{k}}$, and so we have $w^{-1}tw = v^{-1}uu^{-1}suu^{-1}v = v^{-1}sv \in A$. Also, we have

$$l_p(w^{-1}tw) = l_p(v^{-1}sv) = 2l_p(v) + l_p(s) - 2(\mathbf{m} - 2\mathbf{k}) = 2l_p(uw) + l_p(s) - 2(\mathbf{m} - 2\mathbf{k}),$$

but $w \in W_{1,u,\mathbf{k}}$ so $l_p(uw) = l_p(u) + l_p(w) - 2\mathbf{k}$ so that we get

$$\begin{aligned} l_p(w^{-1}tw) &= 2(l_p(u) + l_p(w) - 2\mathbf{k}) + l_p(s) - 2(\mathbf{m} - 2\mathbf{k}) \\ &= 2l_p(w) + 2l_p(u) + l_p(s) - 2\mathbf{m} \\ &= 2l_p(w) + l_p(t) - 2\mathbf{m}. \end{aligned}$$

Therefore $B_{\mathbf{k}} \subseteq A_{t,\mathbf{m}}$. Similarly, $A_{t,\mathbf{m}} \cap W_{1,u,\mathbf{k}} \subseteq B_{\mathbf{k}}$. We note that $B_{\mathbf{k}} = \emptyset$ unless $2\mathbf{k} \leq \mathbf{m}$, and thus

$$A_{t,\mathbf{m}} = \bigcup_{\mathbf{k}:\mathbf{0} \leq 2\mathbf{k} \leq \mathbf{m}} B_{\mathbf{k}}.$$

So $A_{t,\mathbf{m}} \in \mathscr{B}_p(W)$ since it is a finite union of sets in $\mathscr{B}_p(W)$, finishing the proof of (a).

For the first part of (b), we notice that for $A, B \in \mathscr{B}'_p(T)$ or $A, B \in \mathscr{B}_p(T)$ and $r \in S$, it is clear that $(A \cup B)_r = A_r \cup B_r$, $(A \cap B)_r = A_r \cap B_r$ and $(A^c)_r = (A_r)^c$ so that $\mathscr{B}'_p(T)$ and $\mathscr{B}_p(T)$ are clearly Boolean subalgebras of $\mathscr{P}(T)$ and $\mathscr{B}''_p(T)$

is a Boolean subalgebra by definition. Now, suppose that $A \in \mathscr{B}'_p(T)$. Then $A_r \in \mathscr{B}_p(W)$ by definition. Also, for any $r \in S$ and $\mathbf{m} \in \mathcal{M}$, we have the set

$$\begin{aligned} B_{r,\mathbf{m}} &= \{w \in W |\ l_p(w^{-1}rw) = 2l_p(w) + l_p(r) - 2\mathbf{m}\} \\ &= L(1, 2, \mathbf{m}, (1, 1), (1, -1), (e, r, e), \emptyset, \emptyset, \emptyset) \end{aligned}$$

following notation from section 5.10. Thus, by Lemma 5.10.1 we have $B_{r,\mathbf{m}} \in \mathscr{B}_p(W)$ for all $r \in S$ and $\mathbf{m} \in \mathcal{M}$. Finally, we see that $A_{r,\mathbf{m}} = A_r \cap B_{r,\mathbf{m}}$, and thus $A_{r,\mathbf{m}} \in \mathscr{B}_p(W)$ for all $r \in S$ and $\mathbf{m} \in \mathcal{M}$ so that $A \in \mathscr{B}_p(T)$.

Now, we show that $R^p_{x,\mathbf{m}} \in \mathscr{B}_p(T)$ for all $x \in W$ and $\mathbf{m} \in \mathcal{M}$. For any $r \in S$ and $\mathbf{n} \in \mathcal{M}$ we have

$$\begin{aligned} (R^p_{x,\mathbf{m}})_{r,\mathbf{n}} &= \{w \in W |\ w^{-1}rw \in T_{x,\mathbf{m}};\ l_p(w^{-1}rw) = 2l_p(w) + l_p(r) - 2\mathbf{n}\} \\ &= \{w \in W |\ l_p(xw^{-1}rwx^{-1}) = 2l_p(x) + l_p(w^{-1}rw) - 2\mathbf{m}; \\ &\qquad l_p(w^{-1}rw) = 2l_p(w) + l_p(r) - 2\mathbf{n}\} \\ &= \{w \in W |\ l_p(xw^{-1}rwx^{-1}) = 2l_p(x) + 2l_p(w) + l_p(r) - 2(\mathbf{m} + \mathbf{n}); \\ &\qquad l_p(w^{-1}rw) = 2l_p(w) + l_p(r) - 2\mathbf{n}\} \\ &= L(1, 2, \mathbf{m} + \mathbf{n}, (1, 1), (-1, 1), (x, r, x^{-1}), \emptyset, \emptyset, \emptyset) \cap B_{r,\mathbf{n}} \end{aligned}$$

using notation from the proof of Lemma 5.10.1, and by the result of that lemma, we get that $(R^p_{x,\mathbf{m}})_{r,\mathbf{n}} \in \mathscr{B}_p(W)$. Thus, $R^p_{x,\mathbf{m}} \in \mathscr{B}_p(T)$ so that $\mathscr{B}''_p(T) \subset \mathscr{B}_p(T)$ as required.

Next, suppose that $A \in \mathscr{B}'_p(T)$. Then $A_r \in \mathscr{B}_p(W)$ for all $r \in S$. Let $w \in W$

be fixed. Then $wAw^{-1} \in \mathscr{B}'_p(T)$ since for $r \in S$ we have

$$\begin{aligned}(wAw^{-1})_r &= \{u \in W \mid u^{-1}ru \in wAw^{-1}\} = \{u \in W \mid w^{-1}u^{-1}ruw \in A\} \\ &= \{vw^{-1} \in W \mid v^{-1}rv \in A\} = \{v \in W \mid v^{-1}rv \in A\}w^{-1} = A_r w^{-1}\end{aligned}$$

is in $\mathscr{B}_p(W)$ because $\mathscr{B}_p(W)$ is closed under multiplication by elements of W.

For $A \in \mathscr{B}_p(T)$ and $w \in W$, we see that for $r \in S$ and $\mathbf{m} \in \mathcal{M}$ we have

$$(wAw^{-1})_{r,\mathbf{m}} = (wAw^{-1})_r \cap \bigcup_{0 \le \mathbf{k} \le l_p(w)} (B(w, r, 2\mathbf{k}) \cap W_{w^{-1},1,\mathbf{k}})w^{-1} \tag{6.3.1}$$

where $B(w, r, 2\mathbf{k}) = \{v \in W \mid l_p(wv^{-1}rvw) = 2l_p(w) + 2l_p(v) + l_p(r) - 2(\mathbf{m} + 2\mathbf{k})\}$. Indeed, if $u \in (wAw^{-1})_{r,\mathbf{m}}$, then $u \in (wAw^{-1})_r$ and $l_p(u^{-1}ru) = 2l_p(u) + l_p(r) - 2\mathbf{m}$. By the previous argument, $u = vw^{-1}$, and then we get $l_p(wv^{-1}rvw^{-1}) = 2l_p(vw^{-1}) + l_p(r) - 2\mathbf{m} = 2(l_p(v) + l_p(w) - 2\mathbf{k}) + l_p(r) - 2\mathbf{m}$ where $\mathbf{k} \le l_p(w)$ (and we note that this means $v \in W_{w^{-1},1,\mathbf{k}}$). Hence, we have inclusion of the left hand side in the right hand side of (6.3.1). However, working the equation backwards, since $\mathbf{k} \le l_p(w)$ we get inclusion the other way proving that $\mathscr{B}_p(T)$ is closed under conjugation.

Suppose that θ is an automorphism of (W, S). Then we see if $A \in \mathscr{B}'_p(T)$ we have for any $r \in S$

$$\begin{aligned}\theta(A)_r &= \{u \in W \mid u^{-1}ru \in \theta(A)\} = \{u \in W \mid \theta^{-1}(u^{-1}ru) \in A\} \\ &= \{u \in W \mid \theta^{-1}(u^{-1})\theta^{-1}(r)\theta^{-1}(u) \in A\} = \{\theta(v) \in W \mid v^{-1}\theta^{-1}(r)v \in A\} \\ &= \theta(A_{\theta^{-1}(r)})\end{aligned}$$

and so $\theta(A)_r \in \mathscr{B}_p(W)$ since $\mathscr{B}_p(W)$ is closed under automorphisms. Notice this holds for any automorphism. For $A \in \mathscr{B}_p(T)$, and θ an automorphism of

(W, S) commuting with p, recall that from the proof of Proposition 5.5.1 that $\theta(l_p(w)) = l_p(w)$. Then for any $r \in S$ and $\mathbf{m} \in \mathcal{M}$ we have

$$\begin{aligned}\theta(A)_{r,\mathbf{m}} &= \{u \in W|\ u^{-1}ru \in \theta(A); l_p(u^{-1}ru) = 2l_p(u) + l_p(r) - 2\mathbf{m}\\ &= \theta(A)_r \cap \{u \in W|\ l_p(u^{-1}ru) = 2l_p(u) + l_p(r) - 2\mathbf{m}\}.\end{aligned}$$

We know that $\theta(A)_r \in \mathscr{B}_p(W)$ from above, and $\{u \in W|\ l_p(u^{-1}ru) = 2l_p(u) + l_p(r) - 2\mathbf{m}\} \in \mathscr{B}_p(W)$ by Lemma 5.10.1. Thus, we have proved (b).

To show (c), we first note that $\{w \in W|\ w^{-1}rw = t\} \in \mathscr{B}_p(W)$ for all $r \in S$ by Proposition 5.7.1. Therefore $\{t\} \in \mathscr{B}'_p(T)$ for all $t \in T$, and hence any finite subset of T is p-completely regular as well. Also, if $A = \{wrw^{-1}|\ w \in W\}$ is the conjugacy class of $r \in S$, then for any $s \in A$ we have $A_s = \{w \in W|\ w^{-1}sw \in A\} = W$ if s is conjugate to r and $A_s = \emptyset$ if s is not conjugate to r. Since $W \in \mathscr{B}_p(W)$ and $\emptyset \in \mathscr{B}_p(W)$, then $A_s \in \mathscr{B}_p(W)$ for all $s \in S$ and so $A \in \mathscr{B}'_p(T)$.

Finally, we show that (d) is true. We have to show that $B_{\mathbf{m}} := \eta^{-1}(A) \cap X_{\mathbf{m}}$ is regular for all $\mathbf{m} \in \mathcal{M}$ if A is p-completely regular. If $x \in \eta^{-1}(A)$ then $x = rx'$ for some $r \in S$ and $\eta(x) = \nu(x')^{-1}r\nu(x')$. Thus, $\eta^{-1}(A) = \cup_{r\in S}\{r\}B'_r$ where $B'_r = \{x \in S^*|\ \nu(x)^{-1}r\nu(x) \in A\}$. Since $x \in X_{\mathbf{m}}$, we also know that $l_p(\eta(x)) = l_p(\nu(x')^{-1}r\nu(x')) = 2\ell_p(x') + l_p(r) - 2\mathbf{m}$. Therefore, we see that $B_{\mathbf{m}} = \cup_{r\in S} rB_{r,\mathbf{m}}$ where

$$B_{r,\mathbf{m}} := \{x' \in S^*|\ \nu(x')^{-1}r\nu(x') \in A;\ l_p(\nu(x')^{-1}r\nu(x')) = 2\ell_p(x') + l_p(r) - 2\mathbf{m}\}.$$

Hence, to show that $B_{\mathbf{m}}$ is regular, we need to show that each $B_{r,\mathbf{m}}$ is regular (as $B_{\mathbf{m}}$ is then a finite union of regular sets).

Now, if $x' \in S^*_{\mathbf{k}}$ we know that $\ell_p(x') = l_p(\nu(x')) + 2\mathbf{k}$. Therefore, we know that

$B_{r,\mathbf{m}} \cap S^*_{\mathbf{k}}$ is given by the set

$$\{x' \in S^* | \nu(x')^{-1} r \nu(x') \in A;\ l_p(\nu(x')^{-1} r \nu(x')) = 2l_p(x') + l_p(r) - 2(\mathbf{m} - 2\mathbf{k})\},$$

which is equal to $\nu^{-1}(A_{r,\mathbf{m}-2\mathbf{k}}) \cap S^*_{\mathbf{k}}$. Now, we know that since A is p-completely regular in T then $A_{r,\mathbf{n}} \in \mathscr{B}_p(W)$ for all $\mathbf{n} \in \mathcal{M}$ so that $A_{r,\mathbf{m}-2\mathbf{k}}$ is p-completely regular in W for all $\mathbf{k}$. Thus, by Proposition 5.5.1 we know that $\nu^{-1}(A_{\mathbf{m}-2\mathbf{k}}) \cap S^*_{\mathbf{k}}$ is regular. Also, we see that $A_{r,\mathbf{m}-2\mathbf{k}} = \emptyset$ unless $\mathbf{0} \leq 2\mathbf{k} \leq \mathbf{m}$. Therefore we have

$$B_{r,\mathbf{m}} = \cup_{\mathbf{k} \in \mathcal{M}} (B_{r,\mathbf{m}} \cap S^*_{\mathbf{k}}) = \cup_{\mathbf{0} \leq \mathbf{k} \leq 2\mathbf{m}} (B_{r,\mathbf{m}} \cap S^*_{\mathbf{k}})$$

which is a finite union and thus regular, completing the proof of (d). □

6.4 Maps of Boolean Algebras

In this section, we prove some technical facts about Boolean subalgebras that will be useful for describing regularity in mixed products of T and W.

Suppose that $X_1, X_2, ..., X_n$ are sets with each $X_i \neq \emptyset$. Let $X := X_1 \times \cdots \times X_n$. For each i let

$$\pi_i : X \to X_i$$

be the natural projection and let

$$\pi_i^c : X \to X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n$$

be the natural projection where $\hat{X}_i$ means we omit X_i. For each i let $\mathscr{B}_i$ be a Boolean subalgebra of $\mathscr{P}(X_i)$. Then let $\mathscr{B} := \mathscr{B}_1 * \cdots * \mathscr{B}_n$ represent the Boolean subalgebra of $\mathscr{P}(X)$ generated by all $A_1 \times \cdots \times A_n$ with $A_i \in \mathscr{B}_i$.

For any $Y \in \mathscr{P}(X_1) * \cdots * \mathscr{P}(X_n) = \mathscr{P}(X)$, we define an equivalence relation on X_i, $\equiv_{Y,i}$, given by $a \equiv_{Y,i} b$ if $\pi_i^c\left(\pi_i^{-1}(a) \cap Y\right) = \pi_i^c\left(\pi_i^{-1}(b) \cap Y\right)$.

Lemma 6.4.1. *Suppose $X = X_1 \times \cdots \times X_n$, $\mathscr{B}_1 * \cdots * \mathscr{B}_n$ are as above, and $Y \in \mathscr{P}(X)$. Then the following hold.*

(a) For each i, there exist finitely many equivalence classes of $\equiv_{Y,i}$, denoted $Y_{1,i}, Y_{2,i}, ..., Y_{n_i,i}$.

(b) Define a rectangle to be a set of the form $Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n}$ (where each set in the product is one of the finitely many equivalence classes in (a)). If a point of Y is in a rectangle, then that rectangle is a subset of Y, and thus, Y is a union of (finitely many) rectangles.

(c) $Y \in \mathscr{B}_1 * \cdots * \mathscr{B}_n$ if and only if $Y_{j,i} \in \mathscr{B}_i$ for all $i \in \{1, .., n\}$ and all $j \in \{1, ..., n_i\}$.

(d) Let F_i be a subset of X_i for each i such that $F_i \cap Y_{j,i} \neq \emptyset$ for all $j \in \{1, ..., n_i\}$. Consider $F' := Y \cap F$ of $F := F_1 \times \cdots \times F_n$, and let $\equiv_{F',i}$ be the corresponding equivalence relation on each F_i (in the same manner as $\equiv_{Y,i}$ on X_i). Then the equivalence classes for $\equiv_{F',i}$ are precisely the intersections $Y_{j,i} \cap F_i$. In particular, the rectangles for $Y \cap F$ that are contained in $Y \cap F$ are the intersections with F of the rectangles for Y which are contained in Y.

Proof. Since $Y \in \mathscr{P}(X)$, we write

$$Y = \bigcup_{k \in K} A_{k,1} \times \cdots \times A_{k,n}$$

where K is a finite index set. Then define C_i to be the Boolean subalgebra of $\mathscr{P}(X_i)$ generated by the finite set of elements $\{A_{k,i} |\ k \in K\}$. For each $k \in K$, let $A^r_{k,i} := A_{k,i}$ and let $A^c_{k,i} = X_i \setminus A_{k,i}$ (the complement). Then for a sequence $\epsilon := (\epsilon_1, ..., \epsilon_{|K|})$ with each $\epsilon_i \in \{r, c\}$ we define $A_{i,\epsilon} := \cap_{k \in K} A^{\epsilon_k}_{k,i}$. There are only finitely many such sequences, and so we call the finite set $\{A_{i,\epsilon} |\ A_{i,\epsilon} \neq \emptyset\}$ the set of atoms of C_i. We necessarily have $A_{i,\epsilon} \cap A_{i,\epsilon'} = \emptyset$ for $\epsilon \neq \epsilon'$, and each set $J \in C_i$ is a finite union of atoms. Suppose now that $a, b \in A_{i,\epsilon}$ for some ϵ. Let $J \subseteq K$ such that $\epsilon_j = r$ for $j \in J$ and $\epsilon_j = c$ for $j \notin J$. Then $\{A_{j,i} |\ a \in A_{j,i}\} = \{A_{j,i} |\ j \in J\} = \{A_{j,i} |\ b \in A_{j,i}\}$. Therefore,

$$\pi^c_i \left(\pi^{-1}_i(a) \cap Y\right) = \bigcup_{j \in J} A_{j,1} \times \cdots \times \hat{A_{j,i}} \times \cdots \times A_{j,n} = \pi^c_i \left(\pi^{-1}_i(b) \cap Y\right);$$

and thus $a \equiv_{Y,i} b$ so that each atom $A_{i,\epsilon}$ is contained in an equivalence class of $\equiv_{Y,i}$. Therefore, the atoms give a finer partition of X_i than $\equiv_{Y,i}$. Since there are only finitely many atoms, there are only finitely many equivalence classes of $\equiv_{Y,i}$ proving (a).

Now, suppose that $Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n}$ is a rectangle and that $(a_1, ..., a_n) \in Y \cap (Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n})$. Let $(b_1, ..., b_n) \in Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n}$, as well. Then, since $a_1 \equiv_{Y,1} b_1$, we know that $(b_1, a_2, ..., a_n) \in Y$. Working inductively, since $a_i \equiv_{Y,i} b_i$ for all i, we have $(b_1, ..., b_i, a_{i+1}, ..., a_n) \in Y$ for all i. In particular, $(b_1, ..., b_n) \in Y$. Therefore, $Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n} \subseteq Y$, and so Y is a finite union of rectangles proving (b).

Next, if $Y_{j_i,i} \in \mathscr{B}_i$ for all $i \in \{1, ..., n\}$ and $j_i \in \{1, ..., n_i\}$ then $Y_{j_1,1} \times Y_{j_2,2} \times \cdots \times Y_{j_n,n} \in \mathscr{B}_1 * \cdots * \mathscr{B}_n$ for all rectangles and since Y is a finite union of rectangles by (b), we have $Y \in \mathscr{B}_1 * \cdots * \mathscr{B}_n$. Conversely, if $Y \in \mathscr{B}_1 * \cdots * \mathscr{B}_n$, then from the proof of (a), each $A_{k,i} \in \mathscr{B}_i$, and so each $A_{i,\epsilon} \in \mathscr{B}_i$ and so $C_i \subseteq \mathscr{B}_i$. However, each $Y_{j_i,i} \in C_i \subseteq \mathscr{B}_i$ as required for (c).

Finally, let F_i, F, and F' be as in the statement of (d). Let $a \equiv_{Y,i} b$ and let

$a, b \in F_i$. Then we know that

$$\pi_i^c\left(\pi_i^{-1}(a) \cap Y\right) = \pi_i^c\left(\pi_i^{-1}(b) \cap Y\right).$$

Since both $a, b \in F_i$, this implies that

$$\pi_i^c\left(\pi_i^{-1}(a) \cap (Y \cap F)\right) = \pi_i^c\left(\pi_i^{-1}(b) \cap (Y \cap F)\right),$$

and thus $a \equiv_{F',i} b$. Therefore $Y_{j,i} \cap F_i \subseteq F'_{k,i}$ for some k.

Now, suppose that $a \equiv_{F',i} b$. Then $a, b \in F_i$ and we have

$$\pi_i^c\left(\pi_i^{-1}(a) \cap (Y \cap F)\right) = \pi_i^c\left(\pi_i^{-1}(b) \cap (Y \cap F)\right).$$

Let $(x_1, ..., a, ..., x_n) \in \pi_i^{-1}(a) \cap Y$. Then, since $Y_{j,m} \cap F_m \neq \emptyset$, we have $x_m \equiv_{Y,m} x'_m$ with $x'_m \in F_m$ for all $m \neq i$. Therefore, we see that $(x'_1, ..., a_i, ..., x'_n) \in \pi_i^{-1}(a) \cap (Y \cap F)$. Since $a \equiv_{F',i} b$, then $(x'_1, ..., b, ..., x'_n) \in \pi_i^{-1}(b) \cap (Y \cap F)$, and using the equivalence $x_m \equiv_{Y,m} x'_m$ in reverse, this implies that $(x_1, ..., b, ..., x_n) \in \pi_i^{-1}(b) \cap Y$. The exact same argument starting with b instead of a shows

$$\pi_i^c\left(\pi_i^{-1}(a) \cap Y\right) = \pi_i^c\left(\pi_i^{-1}(b) \cap Y\right)$$

so that $a \equiv_{Y,i} b$, and thus $F_{k,i} \subseteq Y_{j,i} \cap F$ for some j. Therefore, we have equality of equivalence classes. Equality of rectangles follows directly. □

Next, we assume we additionally have another family of sets $X'_1, ..., X'_n$. Let $X' := X'_1 \times \cdots \times X'_n$, and suppose that we are given a subset $H_i \subseteq \mathrm{Hom}(X_i, X'_i) \times \mathscr{B}_i$. For each i, we define a Boolean algebra $\mathscr{B}'_i$ (which depends on H_i) of $\mathscr{P}(X'_i)$ consisting of sets $A \subseteq X'_i$ such that for all $(h_i, B_i) \in H_i$ we have $h_i^{-1}(A) \cap B_i \in \mathscr{B}_i$.

Then, let $\mathscr{B}' := \mathscr{B}'_1 * \cdots * \mathscr{B}'_n$ be the corresponding Boolean subalgebra of $\mathscr{P}(X')$. Define $H := \{((h_1 \times \cdots \times h_n), (B_1 \times \cdots \times B_n))|\ (h_i, B_i) \in H_i;\ \forall i\}$. Additionally, we set

$$\mathscr{B}'' := \{Z \subseteq X' |\ h^{-1}(Z) \cap B \in \mathscr{B},\ \text{for all } (h, B) \in H\}.$$

Lemma 6.4.2. *Suppose that for all $i \in \{1, ..., n\}$ there exists a finite subset $H'_i \subset H_i$ such that $\cup_{(h_i,B_i)\in H'_i} h_i(B_i) = X_i$. Then $\mathscr{B}'' \subseteq \mathscr{P}(X'_1) * \cdots * \mathscr{P}(X'_n)$.*

Proof. Define $H' := \{((h_1 \times \cdots \times h_n), (B_1 \times \cdots \times B_n))|\ (h_i, B_i) \in H'_i;\ \forall i\} \subseteq H$. Let $Z \in \mathscr{B}''$. Clearly we have

$$\bigcup_{(h,B)\in H'} h\left(h^{-1}(Z) \cap B\right) \subseteq Z, \tag{6.4.1}$$

and since each $h^{-1}(Z) \cap B \in \mathscr{P}(X)$ and $h : \mathscr{P}(X) \to \mathscr{P}(X')$, we have that the left hand side of (6.4.1) is in $\mathscr{P}(X')$. Now, suppose that $(z_1, ..., z_n) \in Z$. By assumption, we can write $z_i = h_i(b_i)$ for some $(h_i, B_i) \in H'_i$ with $b_i \in B_i$. Then, let $h := h_1 \times \cdots \times h_n$ and $B := B_1 \times \cdots \times B_n$ so that $(h, B) \in H'$ by construction. Then, $b := (b_1, ..., b_n) \in h^{-1}(Z) \cap B$ and so $(z_1, ..., z_n) \in h\left(h^{-1}(Z) \cap B\right)$ for some $(h, B) \in H'$. Therefore, we have

$$Z \subseteq \bigcup_{(h,B)\in H'} h\left(h^{-1}(Z) \cap B\right),$$

and thus $Z \in \mathscr{P}(X')$ since we have equality in (6.4.1). □

Lemma 6.4.3. *With notation as above, the following hold.*

1. *$\mathscr{B}' \subseteq \mathscr{B}''$.*

2. *Suppose that for all $i \in \{1, ..., n\}$ and for all $h \in H_i$ we have that h is*

surjective; furthermore, assume that for each h_i, we have

$$\bigcup_{B_i\colon\ (h_i,B_i)\in H_i} B_i = X_i. \tag{6.4.2}$$

Then $\mathscr{B}'' \subseteq \mathscr{B}'$ (and thus $\mathscr{B}' = \mathscr{B}''$).

Proof. We begin by showing 1. Let $C_i' \in \mathscr{B}_i'$. Then if $(h_i, B_i) \in H_i$ we have $h_i^{-1}(C_i') \cap B_i \in \mathscr{B}_i$. So we have that

$$\begin{aligned}
&(h_1 \times \cdots \times h_n)^{-1}(C_i' \times \cdots \times C_n') \cap (B_1 \times \cdots \times B_n) \\
&= (h_1^{-1}(C_1') \cap B_1) \times \cdots \times (h_n^{-1}(C_n') \cap B_n) \in \mathscr{B}_1 * \cdots * \mathscr{B}_n = \mathscr{B}.
\end{aligned}$$

Therefore, $C_1' \times \cdots \times C_n' \in \mathscr{B}''$. Since $\mathscr{B}''$ is closed under finite unions and intersections, 1 follows.

Now, suppose that for all i and for all $h \in H_i$, we have that h is surjective. Let $Z \in \mathscr{B}''$. According to Lemma 6.4.2, $Z \in \mathscr{P}(X')$, and thus, by Lemma 6.4.1, we have

$$Z = \bigcup_{k\in K} Z_{k,1} \times \cdots \times Z_{k,n}$$

where K is a finite set, and each $Z_{k,1} \times \cdots \times Z_{k,n}$ is a rectangle (in the terminology of 6.4.1). Now, let $(h, B) \in H$ with $h = (h_1 \times \cdots \times h_n)$. We have,

$$V := h^{-1}(Z) = \bigcup_{k\in K} h_1^{-1}(Z_{k,1}) \times \cdots \times h_n^{-1}(Z_{k,n}).$$

A quick calculation shows that if $a \in X_i$ then

$$\pi_i^c\left(\pi_i^{-1}(a) \cap h^{-1}(Z)\right) = h^{-1}\left(\pi_i^c\left(\pi_i^{-1}(h_i(a)) \cap Z\right)\right). \tag{6.4.3}$$

Using equation (6.4.3) and the assumption that all h_i are surjective, then if $a \equiv_{V,i} b$, we get $h_i(a) \equiv_{Z,i} h_i(b)$ and if $a \equiv_{Z,i} b$, we get $a' \equiv_{V,i} b'$ for all $a' \in h_i^{-1}(a)$ and $b' \in h_i^{-1}(b)$. Therefore, each $h_1^{-1}(Z_{k,1}) \times \cdots \times h_n^{-1}(Z_{k,n})$ is a rectangle for $h^{-1}(Z)$ (recalling the definition of a rectangle from 6.4.1).

Now, fix $h = (h_1 \times \cdots \times h_n)$. Consider the sets $\mathbf{Q}_i := \{D_i^{(j)} | \ (h_i, D_i^{(j)}) \in H_i\}$. By assumption

$$h^{-1}(Z) \cap (D_1^{(j_1)} \times \cdots \times D_n^{(j_n)}) \in \mathscr{B}$$

if $D_i^{(j_i)} \in \mathbf{Q}_i$ for all i. Then, we let $\tilde{B}_i = \cup_{j \in J} D_i^{(j)}$ with J finite and all $D_i^{(j)} \in \mathbf{Q}_i$. Since these unions are finite, we have

$$Z' := h^{-1}(Z) \cap (\tilde{B}_1 \times \cdots \times \tilde{B}_n) \in \mathscr{B}.$$

Choose each $\tilde{B}_i$ large enough so that $\tilde{B}_i \cap V_{j,i} \neq \emptyset$ for all $j \in \{1, ..., n_i\}$ (all equivalence classes of $\equiv_{V,i}$); we note that this is possible by our assumption (6.4.2). Then, we have

$$Z' = \bigcup_{k \in K} (h_1^{-1}(Z_{k,1}) \cap \tilde{B}_1) \times \cdots \times (h_n^{-1}(Z_{k,n}) \times \tilde{B}_n) \in \mathscr{B}.$$

By Lemma 6.4.1 (d), we know that $(h_1^{-1}(Z_{k,1}) \cap \tilde{B}_1) \times \cdots \times (h_n^{-1}(Z_{k,n}) \times \tilde{B}_n)$ is a rectangle. Hence, by Lemma 6.4.1 (c), we have that each $h_i^{-1}(Z_{k,i}) \cap \tilde{B}_i \in \mathscr{B}_i$. Finally, since $B_i \in \mathscr{B}_i$ for all i, we know that

$$h_i^{-1}(Z_{k,i}) \cap B_i = h_i^{-1}(Z_{k,i}) \cap \tilde{B}_i \cap B_i \in \mathscr{B}_i.$$

Therefore, for all $i \in \{1, ..., n\}$ and $k \in K$, we have $Z_{i,k} \in \mathscr{B}_i'$ and thus $Z \in \mathscr{B}'$ as required. $\square$

6.5 Regularity in Mixed Products

We now extend our notions of p-complete regularity on T and W to the notion of a p-complete regularity of products $W^n \times T^m$ with $n, m \in \mathbb{N}$.

As done previously when dealing with Cartesian products, we say that a subset of a product $B_1 \times \cdots \times B_n$ with each B_i equal to T or W is p-completely regular if it is in the Boolean subalgebra of $\mathscr{P}(B_1 \times \cdots \times B_n)$ generated by products $A_1 \times \cdots \times A_n$ where each A_i is p-completely regular in W if $B_i = W$ and each A_j is p-completely regular in T for $B_j = T$. To avoid trivial situations, if $n = 0$, we let $B_1 \times \cdots \times B_n$ be the singleton set and declare all subsets to be p-completely regular. For ease of notation in the next following lemma, we assume that $B = B_1 \times \cdots \times B_n = T^k \times W^{n-k}$ with $B_i = T$ for $i \in \{1, ..., k\}$ and $B_i = W$ otherwise.

Lemma 6.5.1. *Let $B = B_1 \times \cdots \times B_n$ be as above, with each $B_i = T$ for $1 \leq i \leq k$ and $B_i = W$ for $k < i \leq n$. For any family $\bar{t} = (t_1, ..., t_k)$ in T^k, we define $f_{\bar{t}} : W^n \to B_1 \times \cdots \times B_n$ by $f_{\bar{t}}(w_1, ..., w_n) = (u_1, ..., u_n)$ where*

$$u_i := \begin{cases} w_i^{-1} t_i w_i; & \text{if } 1 \leq i \leq k \\ w_i; & \text{otherwise.} \end{cases}$$

Then a subset $X \subseteq B_1 \times \cdots \times B_n$ is p-completely regular if and only if for all $\bar{s} = (s_1, ..., s_k) \in S^k$ and $(\mathbf{m}_1, ..., \mathbf{m}_k) \in \mathcal{M}^k$, $f_{\bar{s}}^{-1}(X) \cap \left(T_{s_1,\mathbf{m}_1} \times \cdots \times T_{s_k,\mathbf{m}_k} \times W^{n-k}\right)$ is p-completely regular in W^n.

Proof. This lemma follows from the lemmas in Section 6.4 because the map and sets satisfy the following. Using terminology from that section, we have $H_i' := \cup_{\bar{s} \in S^k} (f_{\bar{s}}, T_{\bar{s},\mathbf{0}})$, which is finite. Since each conjugacy class is p-completely regular, and T is a finite union of conjugacy classes, we reduce to the case where we replace

$B_i = T$ by a conjugacy class in T so that f_{s_i} is surjective. Also, for any $\bar{s} \in S^k$, we have $\cup_{\mathbf{m} \in \mathcal{M}} T_{s_i,\mathbf{m}}$ equal to the conjugacy class of s_i, so we satisfy (6.4.2). Finally, by definition $A \subseteq T$ is p-completely regular in T if and only if $f_s^{-1}(A) \cap T_{s,\mathbf{m}} = A_{s,\mathbf{m}}$ is p-completely regular in W for all $s \in S$ and $\mathbf{m} \in \mathcal{M}$. □

Remark 6.5.2. We see that the above proof would work for mixed products $B = B_1 \times \cdots \times B_n$. We simply used the appropriate ordering for ease of notation in the statement.

Remark 6.5.3. Let $B \cong W^{n-k} \times T^k$ and $B' \cong W^{n'-k'} \times T^{k'}$ be products as above, so that $B \times B' \cong W^{n+n'-k-k'} \times T^{k+k'}$ is another such product. Let R be a p-completely regular subset of $B \times B'$. Then for $b' \in B'$ the set $B_{b',R} := \{b \in B | \ (b, b') \in R\}$ is p-completely regular in B. Also $\{b \in B | \ (b, b') \in R$ for some $b' \in B'\}$ and $\{b \in B | \ (b, b') \in R$ for all $b' \in B'\}$ are both p-completely regular.

6.6 General Regularity in Mixed Products

Now, we generalize Lemma 5.10.1 to the case of mixed products of copies of W and T.

Consider $B = B_1 \times \cdots \times B_n \cong W^{n-k} \times T^k$ as above. Then, we let F be the free group on n generators $X_1, ..., X_n$. For any element $b = (b_1, ..., b_n) \in B$, we define a group homomorphism $\alpha_b : F \to W$ determined by $\alpha_b(X_i) = b_i$. Now, fix an element $g \in F$, and suppose $g = X_{i_1}^{n_1} \cdots X_{i_m}^{n_m}$ is a reduced word for g with $n_i \in \mathbb{Z}$ for all i. Then $\alpha_b(g) = b_{i_1}^{n_1} \cdots b_{i_m}^{n_m}$. Then, we can define a length function on F, dependent on b, by $l_p^b : F \to \mathcal{M}$ by $l_p^b(g) = \sum_{j=1}^m |n_j| l_p(b_{i_j})$ where g is as above and $|n_j|$ represents the absolute value. Now, we can see that $l_p^b(g) - l_p(\alpha_b(g)) \in 2\mathcal{M}$. So for any $g \in F$ and $b \in B$, we call the quantity $l_p^b(g) - l_p(\alpha_b(g))$ the g-deficiency of b.

Theorem 6.6.1. *For any* $\mathbf{m} \in \mathcal{M}$ *and* $g \in F$*, the set*

$$B_{g,\mathbf{m}} := \{b \in B | \ l_p(\alpha_b(g)) = l_p^b(g) - 2\mathbf{m}\}$$

is a p*-completely regular subset of* B*.*

Proof. Fix $\mathbf{m} \in \mathcal{M}$ and $g \in F$ with $g = X_{i_1}^{n_1} \cdots X_{i_m}^{n_m}$. Let $D := \{i_1, ..., i_m\}$. Then let $C := \{j_1, ..., j_q\} \subset D$ be such that $B_j = T$ for all $j \in C$ and $B_j = W$ for all $j \in D \setminus C$. Then for any $\bar{r} \in S^k$, we consider $W_{g,\mathbf{m}}^{\bar{r}} := f_{\bar{r}}^{-1}(B_{g,\mathbf{m}})$. Let $Z_i := W$ if $B_i = W$ and $Z_i = T_{r_i,\mathbf{n}_i}$ otherwise, and let $(\mathbf{n}_1, ..., \mathbf{n}_k) \in \mathcal{M}^k$. For $w := (w_1, ..., w_n) \in W_{g,\mathbf{m}}^{\bar{r}} \cap (Z_1 \times \cdots \times Z_n)$, we have

$$v_w := x_{i_1}^{n_1} \cdots x_{i_m}^{n_m}$$

where $x_{i_k} = w_{i_k}$ if $i_k \notin D$ and $x_{i_k} = w_{i_k}^{-1} r_{i_k} w_{i_k}$ if $i_k \in D$. Thus, we have

$$K_{v_w} := \sum_{i_k : i_k \in D} (2|n_{i_k}| l_p(w_{i_k}) + 1) + \sum_{i_k : i_k \notin D} (|n_{i_k}| l_p(w_{i_k}))$$

Then, since $\bar{r}$ is fixed, Lemma 5.10.1 proves that

$$W_{g,\mathbf{m}}^{\bar{r}} \cap (Z_1 \times \cdots \times Z_n) = \{(w_1, ..., w_n) \in W^n | \ l_p(v_m) = K_{v_m} - 2\mathbf{m}\} \cap (Z_1 \times \cdots \times Z_n)$$

is p-completely regular. Therefore, by Lemma 6.5.1, since $W_{g,\mathbf{m}}^{\bar{r}} \cap (Z_1 \times \cdots \times Z_n)$ is p-completely regular for all $\bar{r} \in S^k$ and $(\mathbf{n}_1, ..., \mathbf{n}_k) \in \mathcal{M}^k$, then we have $B_{g,\mathbf{m}}$ is p-completely regular. □

6.7 Multivariate Poincaré Series

Recall the notation from 5.6. In [8], it is shown that the Poincaré series for the set of reflections, $P(T;W)$, is rational. Due to the results of this chapter, we obtain the stronger result that the multivariate Poincaré series of the set of reflections as well as any p-completely regular subset of T will be rational.

Furthermore, if we have a mixed product $B := B_1 \times \cdots B_{m+n} \cong W^m \times T^n$, let $\mathcal{M}_1, ..., \mathcal{M}_{m+n}$ be the corresponding multiplicative monomials for each B_i generated by $\mathbf{X}_1, ..., \mathbf{X}_{m+n}$ with corresponding length functions $l_{p_i} : B_i \to \mathcal{M}_i$. Let $\mathbb{Z}[[\mathbf{X}_1, ..., \mathbf{X}_{m+n}]]$ be the completion of the polynomial ring $\mathbb{Z}[\mathbf{X}_1, ..., \mathbf{X}_{m+n}]$. Then according to the transfer matrix method ([2] or [28]), for any p-completely regular subset of $A \subseteq B$, we have that the multivariate Poincaré series

$$P(A;B) = \sum_{(x_1,...,x_{m+n}) \in A} l_{p_1}(x_1) \cdots l_{p_{m+n}}(x_{m+n})$$

is rational. In fact, $P(A;B)$ is a finite sum of products, each of the form $f_1 \cdots f_{m+n}$ with each f_i a rational expression in the variables X_i, i.e. $f_i \in \mathbb{Q}(X_i)$.

Example 6.7.1. Let $l_p : W \to \mathcal{M}$, $l_{p_1} : W \to \mathcal{M}_1$, $l_{p_2} : W \to \mathcal{M}_2$, and $l_{p_3} : T \to \mathcal{M}_3$ be generalized length functions with $\mathcal{M}$ considered additively and $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ considered multiplicatively.

1. According to Theorem 6.6.1, for $\mathbf{n} \in \mathcal{M}$,

$$\text{Add}(\mathbf{n}) := \{(x,y) \in W \times W |\ l_p(xy) = l_p(x) + l_p(y) - 2\mathbf{n}\}$$

is p-completely regular in $W \times W$. Therefore,

$$P(\text{Add}(\mathbf{n}); W \times W) = \sum_{(x,y)\in \text{Add}(\mathbf{n})} l_{p_1}(x) l_{p_2}(y) = \sum_{\mathbf{x}\in\mathcal{M}_1;\mathbf{x}'\in\mathcal{M}_2} a_{x,x'}\mathbf{x}\mathbf{x}'$$

has a rational expression, where $a_{x,x'} \in \mathbb{N}$ is the number of pairs $(x,y) \in W \times W$ with $l_p(xy) = l_p(x) + l_p(y) - 2\mathbf{n}$ and $l_{p_1}(x) = \mathbf{x}$ and $l_{p_2}(y) = \mathbf{x}'$.

2. Also by Theorem 6.6.1, for $\mathbf{n} \in \mathcal{M}$,

$$\mathbf{U_n} := \{(x,t) \in W \times T |\ l_p(xtx^{-1}) = 2l_p(x) + l_p(t) - 2\mathbf{n}\}$$

is p-completely regular in $W \times T$. It follows that

$$P(\mathbf{U_n}; W \times T) = \sum_{(x,t)\in\mathbf{U_n}} l_{p_1}(x) l_{p_3}(t)$$

has a rational expression. Moreover, according to Remark 6.5.3, for any $x \in W$, the set

$$\mathbf{U_n}(x) := \{t \in T |\ l_p(xtx^{-1}) = 2l_p(x) + l_p(t) - 2\mathbf{n}\}$$

is p-completely regular in T so

$$P(\mathbf{U_n}(x); T) = \sum_{t\in\mathbf{U_n}(x)} l_{p_3}(t)$$

has a rational expression.

CHAPTER 7

HECKE ALGEBRA MODULES

The recurrence formulae given in section 3.7 look very similar to recurrence formulae for the generic Iwahori-Hecke Algebra. In this chapter, we will further discuss this algebra, and we introduce a large family of modules for the Iwahori-Hecke algebra. These modules arise from the sets $T_{\leq \mathbf{m}}$. In, [18], Dyer uses modules like these to prove a weak form of Lusztig's conjecture about the boundedness of the **a**-function. There is some hope that deeper regularity properties of the modules described in this section may lead to more results about the **a**-function.

Throughout this chapter, we fix an arbitrary Coxeter system, (W, S), with reflections T, reflection cocycle $N : W \to \mathscr{P}(T)$, and with generalized length function $l_{p,W}$ where p is a surjective function from T to the set of indeterminates X as in section 3.2. In addition to X, let V be another set of indeterminates equipped with a surjective function $q : T \to V$ satisfying $q(t) = q(t')$ if t is conjugate to t'. For ease of notation, we denote $p(r) = p_r$ and $q(r) = q_r$ for any $r \in S$.

7.1 The Generic Iwahori-Hecke Algebra

Let R denote the integral polynomial ring $R = \mathbb{Z}[X, V]$ in indeterminates $X \cup V$. We consider the generic Iwahori-Hecke algebra $\mathscr{H}$ of (W, S) over R with generators (as a unital R-algebra) t_r for $r \in S$ subject to the braid relations of (W, S)

$$\underbrace{t_r t_s \cdots}_{m(r,s) \text{ factors}} = \underbrace{t_s t_r \cdots}_{m(r,s) \text{ factors}}$$

and the quadratic relations $t_r^2 = p_r + q_r t_r$. As an R-module, $\mathscr{H}$ is free with R-basis $\{t_w\}_{w \in W}$. The multiplication is determined by the following formulae: $t_1 = \mathrm{Id}_{\mathscr{H}}$ and

$$t_r t_w = \begin{cases} t_{rw} & \text{if } r \notin N(w) \\ p_r t_{rw} + q_r t_w & \text{if } r \in N(w) \end{cases}.$$

By these formulae, it is clear that the set $\{t_w |\ w \in W\}$ is a spanning set for $\mathscr{H}$ as a $\mathbb{Z}[X, V]$-module. In fact, this set is also a $\mathbb{Z}[X, V]$-basis for $\mathscr{H}$ (see [26]).

7.2 A Module for $\mathscr{H}$

We regard R as a free $\mathbb{Z}[V]$ module with basis corresponding to $\mathcal{M} := \mathcal{M}_X$, i.e. monomials in $\mathbb{Z}[X]$. We will denote monomials in $\mathcal{M}_X$ by boldface letters, e.g. $\mathbf{m}$. For any $\mathbf{m} \in \mathcal{M}$, we denote the dual basis element of $R^\dagger := \mathrm{Hom}_{\mathbb{Z}[V]}(R, \mathbb{Z}[V])$ by $\mathbf{m}^*$ so that $\mathbf{m}^*(\mathbf{n}) = 0$ if $\mathbf{n} \neq \mathbf{m}$ and $\mathbf{m}^*(\mathbf{m}) = 1$.

$R^\dagger$ is an R-module with the standard R-structure given by $(rf)(r') = f(rr')$ for $f \in R^\dagger$ and $r, r' \in R$. Then, we note that

$$p_r \mathbf{m}^* = \begin{cases} \left(\frac{\mathbf{m}}{p_r}\right)^* & \text{if } p_r \leq \mathbf{m} \\ 0 & \text{otherwise} \end{cases}$$

Let R' be the R-submodule of $R^\dagger$ spanned by the dual basis elements of $\mathcal{M}_X$. Then R' is free as a $\mathbb{Z}[V]$-module with basis $\mathcal{M}_X$.

Next, we can form the $\mathscr{H}$-module $\mathscr{H}' := \mathscr{H} \otimes_R R'$. For ease, we denote the element $t_w \otimes \mathbf{m}^*$ of $\mathscr{H}'$ by $t_{w,\mathbf{m}}$ for $\mathbf{m} \in \mathcal{M}$ and $w \in W$. These elements form a $\mathbb{Z}[V]$-basis of $\mathscr{H}'$. We thus have

$$p_r t_{w,\mathbf{m}} = \begin{cases} t_{w,\frac{\mathbf{m}}{p_r}} & \text{if } p_r \leq \mathbf{m} \\ 0 & \text{otherwise} \end{cases}$$

for all $r \in S$ and

$$t_r t_{w,\mathbf{m}} = \begin{cases} t_{rw,\mathbf{m}} & \text{if } r \notin N(w) \\ t_{rw,\frac{\mathbf{m}}{p_r}} + q_r t_{w,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \leq \mathbf{m} \\ q_r t_{w,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \not\leq \mathbf{m} \end{cases}$$

for all $r \in S$.

For each $\mathbf{n} \in \mathcal{M}$, the $\mathbb{Z}[V]$-span of the elements $t_{w,\mathbf{m}}$ with $w \in W$ and $\mathbf{m} \leq \mathbf{n}$ is a $\mathscr{H}$-submodule, called $\mathscr{H}'_{\mathbf{n}}$ of $\mathscr{H}'$. Provided S is finite, then $\mathscr{H}'_{\mathbf{n}}$ has finite rank as a $\mathbb{Z}[V]$-module. For $\mathbf{n} \geq \mathbf{m} > \mathbf{0}$, multiplication by $\mathbf{m}$ induces a short exact sequence of $\mathscr{H}$-modules:

$$0 \longrightarrow \sum_{\mathbf{k}<\mathbf{m}} \mathscr{H}'_{\mathbf{k}} \longrightarrow \mathscr{H}'_{\mathbf{n}} \xrightarrow{\mathbf{m}} \mathscr{H}'_{\frac{\mathbf{n}}{\mathbf{m}}} \longrightarrow 0$$

7.3 The Hecke Algebra Modules from $h_{W,p,\mathscr{M}_\infty}$

For $\mathbf{m} \in \mathcal{M}$, let $Q_{\mathbf{m}} := \{N(w) \cap T_{\leq \mathbf{m}} |\ w \in W\}$. Define a free $\mathbb{Z}[V]$-module $\mathscr{H}''$ with $\mathbb{Z}[V]$-basis given by formal symbols $t_{A,\mathbf{m}}$ with $\mathbf{m} \in \mathcal{M}$ and $A \in Q_{\mathbf{m}}$. For $\mathbf{n} \in \mathcal{M}$, let $\mathscr{H}''_{\mathbf{n}}$ be the $\mathbb{Z}[V]$-submodule spanned by the basis elements $t_{A,\mathbf{m}}$ with $\mathbf{m} \leq \mathbf{n}$ and $A \in Q_{\mathbf{m}}$.

Proposition 7.3.1. *1. There is a unique $\mathscr{H}$-module structure on $\mathscr{H}''$ such that the $\mathbb{Z}[V]$-module map $\rho : \mathscr{H}' \longrightarrow \mathscr{H}''$ that is determined by $\rho(t_{w,\mathbf{m}}) =$*

$t_{N(w)\cap T_{\le \mathbf{m}},\mathbf{m}}$ *is an* $\mathscr{H}$*-module epimorphism.*

2. *For* $\mathbf{m} \in \mathcal{M}$, $\mathscr{H}''_{\mathbf{m}} = \rho(\mathscr{H}'_{\mathbf{m}})$ *is a submodule of* $\mathscr{H}''$*. So* $\mathscr{H}''_{\mathbf{m}}$ *is a submodule of* $\mathscr{H}''_{\mathbf{n}}$ *if* $\mathbf{m} \le \mathbf{n}$.

Proof. We define $Z[V]$-linear endomorphisms p'_r and θ_r on $\mathscr{H}''$ for $r \in S$ by

$$p'_r(t_{A,\mathbf{m}}) = \begin{cases} t_{A\cap T_{\le \frac{\mathbf{m}}{p_r}},\frac{\mathbf{m}}{p_r}} & \text{if } p_r \le \mathbf{m} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\theta_r(t_{A,\mathbf{m}}) = \begin{cases} t_{A',\mathbf{m}} & \text{if } r \notin N(w) \\ t_{A''_r,\frac{\mathbf{m}}{p_r}} + q_r t_{A,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \le \mathbf{m} \\ q_r t_{A,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \not\le \mathbf{m} \end{cases}$$

where

$$A' := (rAr \cup \{r\}) \cap T_{\le \mathbf{m}} \text{ and } A''_r := (rAr \setminus \{r\}) \cap T_{\le \frac{\mathbf{m}}{p_r}}$$

Now suppose that $t_{w,\mathbf{m}} \in \mathscr{H}'$. Then on one hand we have

$$\rho(p_r(t_{w,\mathbf{m}})) = \left\{ \begin{array}{ll} \rho(t_{w,\frac{\mathbf{m}}{p_r}}) & \text{if } p_r \le \mathbf{m} \\ \rho(0) & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{ll} t_{N(w)\cap T_{\le \frac{\mathbf{m}}{p_r}},\frac{\mathbf{m}}{p_r}} & \text{if } p_r \le \mathbf{m} \\ 0 & \text{otherwise} \end{array} \right\}$$

and on the other hand we have that

$$p'_r(\rho(t_{w,\mathbf{m}})) = p'_r(t_{N(w)\cap T_{\le \mathbf{m}},\mathbf{m}}) = \left\{ \begin{array}{ll} t_{N(w)\cap T_{\le \frac{\mathbf{m}}{p_r}},\frac{\mathbf{m}}{p_r}} & \text{if } p_r \le \mathbf{m} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Thus, we see that $p'_r(\rho(t_{w,n})) = \rho(p_r(t_{w,n}))$ and extending by linearity we get $p'_r(\rho(h)) = \rho(p_r(h))$ for all $h \in \mathscr{H}'$.

Next, let $t_{w,\mathbf{m}} \in \mathscr{H}'$ and $t_r \in \mathscr{H}$. Again we compute the following:

$$\rho(t_r(t_{w,\mathbf{m}})) = \begin{cases} \rho(t_{rw,\mathbf{m}}) & \text{if } r \notin N(w) \\ \rho(t_{rw,\frac{\mathbf{m}}{p_r}} + q_r t_{w,\mathbf{m}}) & \text{if } r \in N(w) \text{ and } p_r \leq \mathbf{m} \\ \rho(q_r t_{w,\mathbf{m}}) & \text{if } r \in N(w) \text{ and } p_r \nleq \mathbf{m} \end{cases}$$

$$= \begin{cases} t_{N(rw)\cap T_{\leq \mathbf{m}},\mathbf{m}} & \text{if } r \notin N(w) \\ t_{N(rw)\cap T_{\leq \frac{\mathbf{m}}{p_r}},\frac{\mathbf{m}}{p_r}} + q_r t_{N(w)\cap T_{\leq \mathbf{m}},\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \leq \mathbf{m} \\ q_r t_{N(w)\cap T_{\leq \mathbf{m}},\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \nleq \mathbf{m} \end{cases}$$

by linearity of ρ on $\mathbb{Z}[V]$. Also,

$$\theta_r(\rho(t_{w,\mathbf{m}})) = \theta_r(t_{N(w)\cap T_{\leq \mathbf{m}},\mathbf{m}}) = \begin{cases} t_{B',\mathbf{m}} & \text{if } r \notin N(w) \\ t_{B''_r,\frac{\mathbf{m}}{p_r}} + q_r t_{B,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \leq \mathbf{m} \\ q_r t_{B,\mathbf{m}} & \text{if } r \in N(w) \text{ and } p_r \nleq \mathbf{m} \end{cases}$$

where $B = N(w) \cap T_{\leq \mathbf{m}}$. Then $B' = (rBr \cup \{r\}) \cap T_{\leq \mathbf{m}} = N(rw) \cap T_{\leq \mathbf{m}}$ since $r \notin N(w)$ then $N(rw) = \{r\} \cup rN(w)r$. Finally we have $B''_r = rBr \setminus \{r\} \cap T_{\leq \frac{\mathbf{m}}{p_r}} = N(rw) \cap T_{\leq \frac{\mathbf{m}}{p_r}}$ since $r \in N(w)$ then $N(rw) = rN(w)r \setminus \{r\}$. These all follow from Proposition 3.8.1. Therefore we have shown that $\theta_r(\rho(t_{w,n})) = \rho(t_r(t_{w,n}))$ for all $t_{w,n}$ and so extending by linearity we get $\theta_r(\rho(h)) = \rho(t_r(h))$ for all $h \in \mathscr{H}'$.

Now, ρ is clearly an epimorphism,, and so we thus have an $\mathscr{H}$-module structure on $\mathscr{H}''$ in which p_r acts by p'_r for all $r \in S$, q_r acts via the natural $\mathbb{Z}[V]$-module structure on $\mathscr{H}''$ for all $r \in S$, and t_r acts by θ_r for all $r \in S$. Hence, 1 is true. Then 2 follows from the definition of ρ. □

7.4 The **0**-Hecke Algebra

Now, we regard $\mathbb{Z}[V]$ as the R-module R/XR. Then for any $\mathbf{m} \in \mathcal{M}$ we define $\widetilde{\mathscr{H}''_{\mathbf{m}}} := \mathscr{H}''_{\mathbf{m}}/(\sum_{\mathbf{n}<\mathbf{m}} \mathscr{H}''_{\mathbf{n}})$, which is annihilated by X and so may naturally regarded as a $\widetilde{\mathscr{H}} := \mathscr{H}/X\mathscr{H}$ module. We see that $\widetilde{\mathscr{H}}$ is a free $\mathbb{Z}[V]$-module with $\mathbb{Z}[V]$-basis given by the images in the quotient of elements t_w for $w \in W$. By abuse of notation, we will denote these elements still as t_w. We also see that $\widetilde{\mathscr{H}''_{\mathbf{m}}}$ has a free $\mathbb{Z}[V]$-basis consisting of the images of elements $t_{A,\mathbf{m}}$ for $A \in Q_{\mathbf{m}}$. The $\mathbb{Z}[V]$-algebra structure of $\widetilde{H}$ is determined by

$$t_r t_w = \begin{cases} t_{rw}, & \text{if } r \notin N(w) \\ p_r t_w, & \text{if } r \in N(w) \end{cases}$$

for $r \in S$ and $w \in W$, and the $\widetilde{\mathscr{H}}$-module structure on $\widetilde{\mathscr{H}''_{\mathbf{m}}}$ is determined by

$$t_r t_{A,\mathbf{m}} = \begin{cases} t_{(rAr\cup\{r\})\cap T_{\leq \mathbf{m}},\mathbf{m}} & \text{if } r \notin N(w) \\ p_r t_{A,\mathbf{m}} & \text{if } r \in N(w) \end{cases}$$

for $r \in S$ and $A \in Q_{\mathbf{m}}$.

CHAPTER 8

INITIAL SECTIONS OF REFLECTION ORDERS

In this chapter, we recall the definition due to Dyer (c.f. [11], [12], or [13]) of reflection orders, which are certain total orders of the set of reflections T, and of their initial sections, which are subsets of T that are known to lead to partial orders (twisted Bruhat orders) on W that are similar to Bruhat order. In particular, using the initial section $\emptyset \subseteq T$ we get the Bruhat order on W, and using $T \subseteq T$, we get the reverse Bruhat order on W. In this chapter, we determine all subsets of T that give rise to partial orders on W in the same manner. The subsets of T that have this property are those which are "locally" (at each dihedral reflection subgroup) initial sections; it has been conjectured by Dyer ([12]) that these sets coincide with the initial sections. Reflection orders are closely related to a notion of closure in the positive root system. In Chapter 9, we will examine the relationship between closure in root systems and dominance order on the positive roots. For general references about the properties of reflection orders and initial sections, see [2], [11],[12], and [13].

8.1 Classification of Twisted Bruhat Orders

Let (W, S) be a Coxeter system. Recall that W acts on $\mathscr{P}(T)$ by $w \cdot A = N(w) + wAw^{-1}$ for any $A \subseteq T$ and $w \in W$, where $+$ represents symmetric difference. Now, for any $A \subseteq T$, we follow [11] by defining a directed graph $\Omega_{(W,A)}$ with the vertex set of $\Omega_{(W,A)}$ equal to W and edge set $E_{(W,A)} = \{(tw, w) |\ t \in w \cdot A\}$. We also have the following equivalent definition of $E_{(W,A)}$.

$$\begin{aligned} E_{(W,A)} = \{(tw, w) |\ t \in N(w) + wAw^{-1}\} &= \{(tw, w) |\ w^{-1}tw \in N(w^{-1}) + A\} \\ &= \{(wt', w) |\ t' \in N(w^{-1}) + A\}, \end{aligned}$$

and we will find it convenient to use this description, $E_{(W,A)} = \{(wt, w) |\ t \in N(w^{-1}) + A\}$. For $x \in W$, we have that $t \in w \cdot A$ if and only if $t \in (wx^{-1}) \cdot x \cdot A$, and thus the map $w \mapsto wx^{-1}$ defines an isomorphism $\Omega_{(W,A)} \cong \Omega_{(W,x \cdot A)}$. In addition, for $A \subseteq T$ we can define a length function $l_A : W \to \mathbb{Z}$ in the following way:

$$l_A(v, w) = l(wv^{-1}) - 2\Big|[N(vw^{-1}) \cap v \cdot A]\Big| \in \mathbb{Z}$$

and then set $l_A(w) = l_A(1, w)$. We can define a pre-order $\leq_A$ for any $A \subseteq T$ given by the following: $v \leq_A w$ if and only if there exist $t_1, ..., t_n \in T$ with $w = vt_1...t_n$ such that $t_i \notin [N((vt_1...t_{i-1})^{-1}) + A]$ for all $i = 1, ..., n$.

Remark 8.1.1. The multivariate analog of such single variable length functions, l_A, similar to that of Chapter 3, has not been studied but should be of interest.

Definition 8.1.2. Following [12], a total order on the set T' in a dihedral Coxeter system, (W', S') with $S = \{r, s\}$, is called a dihedral reflection order if it is either $\prec_R$ or $\prec'_R$ where $\prec_R$ and $\prec'_R$ are defined by $r \prec_R rsr \prec_R \cdots \prec_R srs \prec_R s$ and $s \prec'_R srs \prec'_R \cdots \prec'_R rsr \prec'_R r$. Then, for any Coxeter system (W, S) we call a total order $\prec_R$ on T a *reflection order* if the restriction $\prec_R\big|_{W' \cap T}$ is a dihedral reflection order for all dihedral reflection subgroups W' of (W, S) with respect to the canonical generators $\chi(W')$.

Remark 8.1.3. We can also define a reflection order in terms of the root system Φ associated to the Coxeter system. This definition can be found in [2], and we will discuss this definition in more detail in Chapter 9.

Recall from [12] that an initial section of a reflection order is a subset $A \subseteq T$ such that there is a reflection order $\prec_R$ with the property that $a \prec_R b$ for all $a \in A$ and $b \in T \setminus A$. It is shown in [11] that $\leq_A$ is a partial order of W if A is an initial section of a reflection order. Our main result in this chapter, Theorem 8.1.4 below, describes all subsets A of T for which $\leq_A$ is a partial order.

Let $\mathbf{A}_{(W,S)}$ be the set of initial sections of reflection orders of T. Now, we define $\hat{\mathbf{A}}_{(W,S)} = \{A \subseteq T |\ A \cap W' \in \mathbf{A}_{(W',\chi(W'))}\ \forall W' \subseteq W \text{ dihedral}\}$. It has been conjectured by Dyer that $\mathbf{A} = \hat{\mathbf{A}}$. We now come to the main result:

Theorem 8.1.4. *Let (W, S) be any Coxeter system. The following are equivalent:*

1. *$\Omega_{(W,A)}$ is acyclic.*

2. *$\leq_A$ is a partial order.*

3. *$\Omega_{(W,A)}$ has no cycle of length four.*

4. *$A \in \hat{\mathbf{A}}$.*

5. *$l_A(xt) < l_A(x)$ for all $x \in W$ and $t \in N(x^{-1}) + A$.*

8.2 Proof of the Classification

In the following proofs, for any positive root $\alpha \in \Phi^+$, let $t_\alpha \in T$ be the corresponding reflection, and for any reflection $t \in T$, let $\alpha_t \in \Phi^+$ be the corresponding positive root. To begin with we investigate the dihedral case. Suppose (W, S) is dihedral, i.e. $S = \{r, s\}$. There is a bijection between subsets

$A \subseteq T$ and subsets $\Psi \subset \Phi$ such that $\Psi \cup -\Psi = \Phi$ and $\Psi \cap -\Psi = \emptyset$ given by $A = A_\Psi = \{t_\alpha |\ \alpha \in \Psi \cap \Phi^+\}$. We note that $A_{-\Psi} = A_\Psi + T$.

Lemma 8.2.1. *For any $A_\Psi \subseteq T$, $w \cdot A_\Psi = A_{w(\Psi)}$.*

Proof. It is enough to show that this is true for $r \in S$. Now we know that $r \cdot A_\Psi = \{r\} + rA_\Psi r = \{r\} + \{rt_\alpha r|\ \alpha \in \Psi \cap \Phi^+\} = \{r\} + \{t_{r(\alpha)}|\ \alpha \in \Psi \cap \Phi^+\} = \{r\} + \{t_\alpha|\ \alpha \in r(\Psi) \cap r(\Phi^+)\}$. If $\alpha_r \in \Psi$ then $-\alpha_r \in r(\Psi) \cap r(\Phi^+)$ and so $r \in \{t_\alpha|\ \alpha \in r(\Psi) \cap r(\Phi^+)\}$. This implies that $r \cdot A_\Psi = \{t_\alpha|\ \alpha \in r(\Psi) \cap \Phi^+\}$. If $\alpha_r \notin \Psi$ then $r \notin \{t_\alpha|\ \alpha \in r(\Psi) \cap r(\Phi^+)\}$. But $\alpha_r \in r(\Psi)$ so $\{r\} + \{t_\alpha|\ \alpha \in r(\Psi) \cap r(\Phi^+)\} = \{t_\alpha|\ \alpha \in r(\Psi) \cap \Phi^+\}$. Thus, in both cases $r \cdot A_\Psi = \{t_\alpha|\ \alpha \in r(\Psi) \cap \Phi^+\}$. □

Since we are considering (W, S) dihedral, we choose an orientation of the plane spanned by Φ. Then for any root $\alpha \in \Phi$ we can define the **neighbor** of α, denoted $nbr(\alpha)$, to be the root lying directly next to α if we traverse the root system clockwise. Recall that $S = \{r, s\}$. Interchanging r and s if necessary, we assume without loss of generality that $\alpha_r = nbr(-\alpha_s)$. We note that this implies that $nbr(\alpha_s) \notin \Phi^+$.

Lemma 8.2.2. *Let (W, S) be dihedral. Suppose $\alpha \in \Phi^+$ and $\gamma := nbr(\alpha) \in \Phi^+$. Then $t_\alpha t_\gamma = sr$.*

Proof. Let $(rsr...)_n$ denote an element with n simple reflections listed (we note that $l((rsr...)_n)$ is not necessarily n). With this terminology, $t \in T$ can be written $t = (rsr...)_{2i+1}$ or $t = (srs...)_{2i+1}$ for some $i \geq 0$. Under the chosen orientation for Φ^+, if $t_\alpha = (rsr...)_{2i+1}$ then we can write $t_\gamma = (rsr...)_{2i+3}$ so that $t_\alpha t_\gamma = (rsr...)_{2i+1}(rsr...)_{2i+3} = sr$. Otherwise, if $t_\alpha = (srs...)_{2i+1}$ $(i \geq 1)$ then $t_\gamma = (srs...)_{2i-1}$ and so $t_\alpha t_\gamma = sr$. □

Now, given $A = A_\Psi$, we introduce two conditions that a positive system, Γ^+, of Φ can have:

C1: There are roots $\alpha, nbr(\alpha) \in \Gamma^+$ such that $\alpha \notin \Psi$ and $nbr(\alpha) \in \Psi$.

C2: There are roots $\beta, nbr(\beta) \in \Gamma^+$ such that $\beta \in \Psi$ with $nbr(\beta) \notin \Psi$.

Lemma 8.2.3. *Let (W, S) be dihedral and let $A = A_\Psi \subseteq T$.*

1. *If the positive system $r(\Phi^+)$ has condition* **C1***, then there exists a path $1 \to x \to sr$ in $\Omega_{(W,A)}$.*

2. *If the positive system $r(\Phi^+)$ has condition* **C2***, then there exists a path $sr \to x \to 1$ in $\Omega_{(W,A)}$.*

Proof. Replacing A by $A + T$ reverses the orientation of edges in $\Omega_{(W,A)}$ and so 2 follows from 1.

We now prove 1. There are two cases to consider. If $\alpha \neq \alpha_s$, then both α and $\gamma := nbr(\alpha) \in \Phi^+$. So $t_\alpha \notin A$ and $t_\gamma \in A$. Also, since $t_\alpha \notin \{r, s\}$, it follows that $t_\gamma \in N(t_\alpha)$. So we have that $t_\gamma \notin N(t_\alpha) + A$. Thus, we have a path $1 \to t_\alpha \to t_\alpha t_\gamma = sr$, where the last equality follows from Lemma 8.2.2. Now, if $\alpha = \alpha_s$ then $nbr(\alpha) = -\alpha_r$. Since $-\alpha_r \in \Psi$ we know that $\alpha_r \notin \Psi$ which implies $r \notin A$. Also, $s \notin A$ and clearly $r \notin N(s)$. Together, we see that there is a path $1 \to s \to sr$ in $\Omega_{(W,A)}$. □

Lemma 8.2.4. *For (W, S) dihedral, let $A = A_\Psi \subseteq T$. If there are no 4-cycles in $\Omega_{(W,A)}$ then there is no positive system, $\Gamma^+ \subset \Phi$ satisfying* **C1** *and* **C2***.*

Proof. Suppose there is a positive system Γ^+ satisfying **C1** and **C2**. It is clear that $-\Gamma^+$ also satisfies **C1** and **C2**. Since Γ^+ satisfies both **C1** and **C2**, then $w(\Gamma^+)$

will also satisfy both conditions (w with even length respects both conditions and w with odd length interchanges the conditions). Thus, we can find a $w \in W$ such that $r(\Phi^+) = w(\Gamma^+)$ or $r(\Phi^+) = w(-\Gamma^+)$. So we have that $r(\Phi^+)$ satisfies **C1** and **C2** with respect to $A_{w(\Psi)} = w \cdot A_\Psi$. Since $\Omega_{(W,A)}$ is isomorphic to $\Omega_{(W,w\cdot A)}$, we can assume without loss of generality that $r(\Phi^+)$ satisfies **C1** and **C2** with respect to A_Ψ. Thus, Lemma 8.2.3 implies that we have a path $sr \to u \to 1 \to v \to sr$ in $\Omega_{(W,A)}$. □

Lemma 8.2.5. *Let (W,S) be dihedral and $A = A_\Psi \subseteq T$. If there is no positive system, Γ^+, of Φ satisfying* **C1** *and* **C2***, then $A \in \mathbf{A}_{(W,S)}$.*

Proof. Let $A \notin \mathbf{A}_{(W,S)}$. Recall that the only two reflection orders on (W,S) are $\prec_R$ and $\prec'_R$ described in Definition 8.1.2. Since A is not an initial section of either of these orders, without loss of generality (replacing A with $T \setminus A$ if necessary) we can find $t_0 \in A$ and $t_1, t_2 \in T \setminus A$ such that $t_1 \prec_R t_0$ and $t_2 \prec'_R t_0$, i.e. $t_2 \prec_R t_0 \prec_R t_1$.

Now, if (W,S) is finite, we can list all of the reflections and thus can find $t', t'' \in T$ such that $t_2 \preceq_R t' \prec_R t_0 \preceq_R t'' \prec_R t_1$ with $\alpha_{t'} \notin \Psi$ and $nbr(\alpha_{t'}) \in \Psi$, and $\alpha_{t''} \in \Psi$ and $nbr(\alpha_{t''}) \notin \Psi$. Thus Φ^+ satisfies **C1** and **C2**.

If (W,S) is infinite, then $T = T_r \cup T_s$ (disjoint union) where $T_r = \{t \in T | \, r \in N(t)\}$ and $T_s = \{t \in T | \, s \in N(t)\}$. Suppose Φ^+ does not satisfy **C1** and **C2**. We cannot have $t_1, t_2, t_0 \in T_u$ for some $u \in \{r, s\}$ since this would imply, by the reasoning for (W,S) finite, that Φ^+ satisfies **C1** and **C2**. So, we can assume that $t_2, t_0 \in T_r$ and $t_1 \in T_s$ (the case $t_2 \in T_r$ and $t_0, t_1 \in T_s$ is exactly similar). Additionally, since **C1** and **C2** aren't both satisfied by Φ^+, we may assume the following conditions hold (see Figure 8.1):

1. $\alpha_{t_0} = nbr(\alpha_{t_2})$,

2. $t \in T \setminus A$ if $t \preceq_R t_2$,

3. $t \in T \setminus A$ if $t \in T_s$ and $t \preceq_R t_1$,

4. $t \in A$ if $t \in T_r$ and $t_0 \preceq_R t$,

5. $t \in A$ if $t_1 \prec_R t$.

Now, consider the positive system Γ^+ with simple roots α_{t_0} and $-nbr(\alpha_{t_0})$. Then Γ^+ satisfies **C1** using the roots α_{t_2} and $\alpha_{t_0} = nbr(\alpha_{t_2})$. Also, by above, $\alpha_{t_1} \notin \Psi$ and $nbr(\alpha_{t_1}) \in \Psi$ (note that even if $t_1 = s$ this is true since $nbr(\alpha_s) = -\alpha_r \in \Psi$ because by above $\alpha_r \notin \Psi$). Using this, we see that Γ^+ satisfies **C2** using the roots $-\alpha_{t_1}$ and $nbr(-\alpha_{t_1}) = -nbr(\alpha_{t_1})$ (again see Figure 8.1).

□

With the dihedral case taken care of, we proceed to the general case. For the remainder of the chapter, we assume that (W, S) is a general Coxeter system.

Proposition 8.2.6. *For all $w \in W$ and $A \in \hat{\mathbf{A}}_{(W,S)}$, $w \cdot A \in \hat{\mathbf{A}}_{(W,S)}$.*

Proof. It suffices to check the condition for $w = r \in S$. Let W' be a dihedral reflection subgroup. If $r \in W'$, then $(r \cdot A) \cap W' = N_{(W',\chi(W'))}(r) + r(A \cap W')r \in \mathbf{A}_{(W',\chi(W'))}$ by [12]. Now, if $r \notin W'$, then $(r \cdot A) \cap W' = r(A \cap rW'r)r$. However, conjugation by r defines an isomorphism $(W', \chi(W')) \cong (rW'r, r\chi(W')r)$ in this case, and by assumption $A \cap rW'r \in \mathbf{A}_{(rW'r,r\chi(W')r)}$, thus $(r \cdot A) \cap W' \in \mathbf{A}_{(W',\chi(W'))}$.

□

Before, we prove Theorem 8.1.4, we provide one more important proposition, which is proven in [11], but we include a proof here as well.

Proposition 8.2.7. *Let $A \in \hat{\mathbf{A}}_{(W,S)}$, $x \in W$, $t \in T$. Then $l_A(x, xt) > 0$ iff $t \notin N(x^{-1}) + A$.*

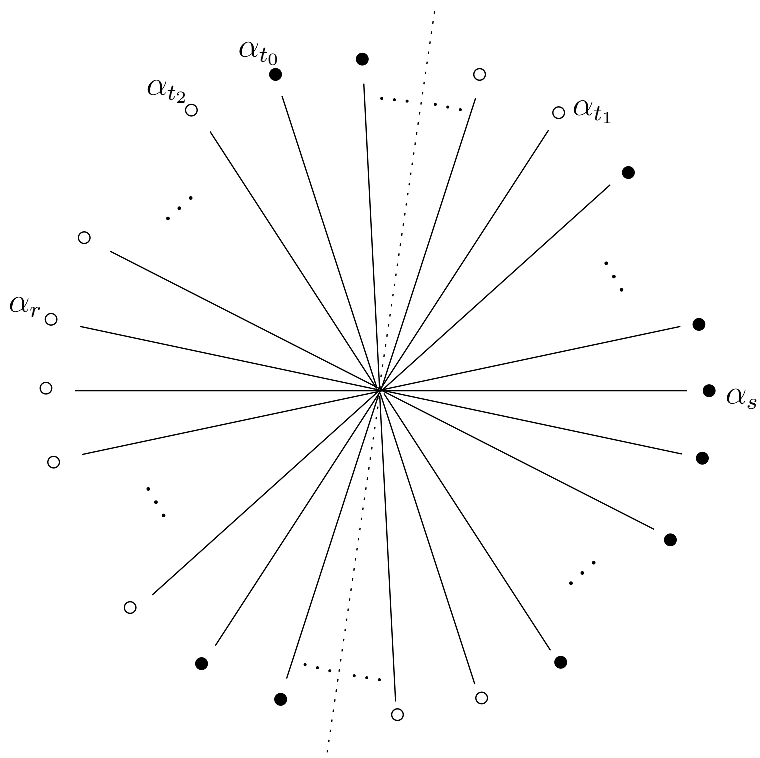


Figure 8.1. This is a schematic diagram of the root system for (W, S) infinite. Φ^+ consists of the roots which are non-negative linear combinations of α_r and α_s. The dotted ray through the origin represents a limit line of roots. If **C1** and **C2** are not both satisfied by Φ^+ then Ψ is pictured above with the roots in Ψ labeled by $\bullet$, and those not in Ψ labeled by $\circ$.

Proof. We begin by proving the statement for all $A \in \hat{\mathbf{A}}_{(W,S)}$ and $t \in T$ with $x = 1$, i.e. $l_A(1,t) = l_A(t) > 0$ if and only if $t \notin A$. If (W,S) is dihedral with $S = \{r,s\}$, then we see that, without loss of generality, $t = (rsr...)_{2n+1}$ with $l(t) = 2n+1$. Then, $N(t) = \{r, rsr, ..., (rsr...)_{2n+1}, ..., (rsr...)_{4n+1}\}$. Since $A \in \hat{\mathbf{A}}_{(W,S)}$, then if $(rsr...)_{2i+1} \in A$ either $(rsr...)_{2j+1} \in A$ for all $j \leq i$ or $(rsr...)_{2j+1} \in A$ for all $i \leq j \leq 2n$. In particular $|N(t) \cap A| \geq n+1$ if and only if $t = (rsr...)_{2n+1} \in A$. Therefore, in this case $l_A(t) > 0$ if and only if $t \notin A$.

Suppose that (W,S) is not dihedral. Then if W' is any dihedral reflection subgroup containing t, since $A \in \hat{\mathbf{A}}_{(W,S)}$ then $A' := A \cap W' \in \hat{\mathbf{A}}_{(W',\chi(W'))}$, and so we have $l_{A'}(t)$ makes sense in the Coxeter system $(W', \chi(W'))$. Now, by equation 2.3.1, we get that

$$N(t) \setminus \{t\} = \dot{\bigcup}_{W' \in \mathscr{M}_t} N_{(W',\chi(W'))}(t) \setminus \{t\}$$

and

$$N(t) \setminus \{t\} \cap A = \dot{\bigcup}_{W' \in \mathscr{M}_t} \left(N_{(W',\chi(W'))}(t) \setminus \{t\}\right) \cap A,$$

which implies that

$$l(t) - 1 = |N(t) \setminus \{t\}| = \sum_{W' \in \mathscr{M}_t} |N_{(W',\chi(W'))}(t) \setminus \{t\}| = \sum_{W' \in \mathscr{M}_t} l_{(W',\chi(W'))}(t) - 1$$

and

$$l_A(t) - 1 = \sum_{W' \in \mathscr{M}_t} (l_{A \cap W'}(t) - 1). \tag{8.2.1}$$

Now, $t \notin A$, if and only if $t \notin A \cap W'$ for all $W' \in \mathscr{M}_t$. This occurs if and only if $l_{A \cap W'}(t) > 0$ for all $W' \in \mathscr{M}_t$ and then by equation (8.2.1) is true if and only if we get $l_A(t) > 0$ as required.

Now, if $x \in W$,

$$l_A(x, xt) = l(xtx^{-1}) - 2|N(xtx^{-1}) \cap x \cdot A|$$
$$= l(xtx^{-1}) - 2|N(xtx^{-1}) \cap 1 \cdot (x \cdot A)| = l_{x \cdot A}(1, xtx^{-1})$$

so $l_A(x, xt) = l_{x \cdot A}(xtx^{-1})$. Therefore, $l_A(x, xt) > 0$ if and only if $l_{x \cdot A}(xtx^{-1}) > 0$.

Also, $t \notin N(x^{-1}) + A$ if and only if $xtx^{-1} \notin xN(x^{-1})x^{-1} + xAx^{-1} = N(x) + xAx^{-1} = x \cdot A$. So $t \notin N(x^{-1}) + A$ if and only if $xtx^{-1} \notin x \cdot A$.

Putting together the previous statements, we have that the statement of the proposition is equivalent to the statement $l_{x \cdot A}(xtx^{-1}) > 0$ if and only if $t \notin x \cdot A$. Following Proposition 8.2.6, we let $A' = x \cdot A \in \hat{\mathbf{A}}_{(W,S)}$ and $t' = xtx^{-1}$, then this case is equivalent to $l_{A'}(t') > 0$ if and only if $t' \notin A'$ which is true due to the $x = 1$ case above. □

At this point, we can use the previous lemmas and propositions to prove Theorem 8.1.4:

Proof. (1)⇔(2)⇒(3) is clear.

(3)⇒(4): According to section 2.3, we know that for any dihedral subgroup W' of W, $N_{(W', \chi(W'))}(w) = N_{(W,S)}(w) \cap W'$. This implies $\Omega_{(W', A \cap W')}$ is a subgraph of $\Omega_{(W,A)}$. So, if $A \notin \hat{\mathbf{A}}_{(W,S)}$, Lemma 8.2.4 and Lemma 8.2.5 imply that there exists some dihedral subgroup W' of W with $\Omega_{(W', A \cap W')}$ containing a cycle with four edges.

(4)⇒(5): Suppose that $A \in \hat{\mathbf{A}}_{(W,S)}$, $x \in W$ and $t \in N(x^{-1}) + A$. Then Proposition 8.2.7 implies that $l_A(x, xt) < 0$. But by [11], $l_A(1, x) + l_A(x, xt) = l_A(1, xt)$ and this implies $l_A(xt) - l_A(x) = l_A(x, xt) < 0$.

(5)⇒(1): Suppose $\Omega_{(W,A)}$ has a cycle. This means that $xt_1...t_n = x$ for some $n > 0$ (all $t_i \in T$ and with $t_i \notin N((xt_1...t_{i-1})^{-1}) + A$ for all i). By assumption, $l_A(x) <$

$l_A(xt_1) < l_A(xt_1t_2) < ... < l_A(xt_1t_2...t_n) = l_A(x)$ which is a contradiction. □

Remark 8.2.8. In case (W, S) is finite, say of order m, we can give a much simpler proof of the equivalence of (1), (2), (4), and (5). By the proof above, we have (4)⇒(5)⇒(1)⇔(2). If (4) fails, then $w \cdot A \neq T$ for all $w \in W$ (or else $A = w^{-1} \cdot T$ which is an initial section). So we can choose $t \notin w \cdot A$. It follows that we can recursively choose $t_1, ..., t_m$ such that $t_1 \notin A$ and $t_i \notin t_{i-1} \cdots t_1 \cdot A$ for $i = 2, ..., m$. This gives us the following path in $\Omega_{(W,A)}$: $1 \to t_1 \to \ldots \to t_m \cdots t_1$. However, this path has $m + 1$ elements and W has m elements, so there must be two elements in the path that are the same thus creating a cycle.

CHAPTER 9

CLOSURE IN THE ROOT SYSTEM

This chapter recalls a notion of closure on the positive roots that has been studied by Dyer [16]. A conjectural relation, due to Dyer, between these closures and the sets $T_{\leq m}$ ($m \in \mathbb{N}$), as defined in section 2.4 and generalized in Chapter 3, would lead to a new and potentially interesting normal form for elements of Coxeter groups, via known special cases of other conjectures of Dyer ([17]) involving initial sections and these closures. The results of the previous chapter afford the possibility of bringing the study of twisted Bruhat orders to bear on these conjectures.

9.1 2-closure

Let (W, S) be a fixed Coxeter system. Recall that we denote the roots and positive roots by Φ and Φ^+ respectively. We begin by describing a notion of closure on the subsets of positive roots, Φ^+. This closure is similar to the notion of closure in the crystallographic root system of a finite Weyl group as described in [3].

Definition 9.1.1. Let $\Gamma \subseteq \Phi^+$. We say that Γ is 2-closed if for all $\alpha, \beta \in \Gamma$, we have $(\mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}\beta) \cap \Phi^+ \subseteq \Gamma$.

Remark 9.1.2. Let $T_\Gamma \subseteq T$ be the subset of reflections corresponding to Γ via the canonical bijection. Then, 2-closure is the same as the following. T_Γ is 2-closed if for all $t_\alpha, t_\beta \in T_\Gamma$ with $W' = \langle t_\alpha, t_\beta \rangle$, $\{t \in W' \cap T |\ t_\alpha \prec_R t \prec_R t_\beta\} \cup \{t \in W' \cap T |\ t_\beta \prec_R t \prec_R t_\alpha\} \subset T_\Gamma$ for all reflection orders $\prec_R$ on T (see Chapter 8).

Definition 9.1.3. We say a subset $\Gamma \subseteq \Phi^+$ is biclosed if Γ and $\Phi^+ \setminus \Gamma$ are both 2-closed.

In particular, initial sections of reflection orders are biclosed subsets. Moreover, $\hat{\mathbf{A}}_{(W,S)}$ as defined in Chapter 8 is the set of all subsets of T corresponding to biclosed subsets of Φ^+ under the natural bijection $\Phi^+ \leftrightarrow T$.

9.2 More types of subsets of Φ^+

We introduce three more types of subsets of Φ^+. These notions are due to Dyer, and further information about them can be found in [17].

Definition 9.2.1. Let $\Gamma \subseteq \Phi^+$.

1. We say Γ is balanced if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\beta \in \Phi^+_{W'}$ such that $l_{(W',\chi(W'))}(s_\beta) < l_{(W',\chi(W'))}(s_\alpha)$ implies that $\beta \in \Gamma$.

2. We say Γ is bipedal if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$ with $\alpha \notin \Pi_{W'}$, $\Pi_{W'} \subset \Gamma$.

3. We say Γ is unipedal if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\Pi_{W'} = \{\beta, \gamma\}$ implies that either $\beta \in \Gamma$ or $\gamma \in \Gamma$.

The following proposition is then clear from the definitions.

Proposition 9.2.2. *Suppose* $\Gamma \subseteq \Phi^+$*.*

1. If Γ *is balanced then* Γ *is bipedal.*

2. *If* Γ *is bipedal, then* Γ *is unipedal.*

3. *If* $\Phi^+ \setminus \Gamma$ *is closed, then* Γ *is unipedal.*

4. *If* $\Gamma \subseteq \Phi^+$ *is bipedal and* $\Delta \subseteq \Phi^+$ *is unipedal, then* $\Gamma \cap \Delta$ *is unipedal.*

For any subset $\Gamma \subseteq \Phi^+$, we let $\bar{\Gamma}$ denote the 2-closure of Γ, that is the smallest 2-closed set containing Γ. The following is conjectured by Dyer in [17].

Conjecture 9.2.3. If $\Gamma \subseteq \Phi^+$ is unipedal, then $\bar{\Gamma}$ is biclosed.

This would strengthen the following conjecture, also in [17], parts of which were raised in [13, Remark 2.1.4].

Conjecture 9.2.4. Let $A := \{\Gamma \subseteq \Phi^+ | \ \Gamma \text{ is biclosed}\}$. Then A is a complete lattice with $\bigvee_{i \in I} \Gamma_i = \overline{\bigcup_{i \in I} \Gamma_i}$ for an arbitrary family $\{\Gamma_i\}_{i \in I} \subset A$.

The last conjecture extends the known fact that W under weak order is a meet semi-lattice (see [2] for example). In fact, [17] proves the following:

Theorem 9.2.5. *Let* $\Phi_w := \Phi^+ \cap w(-\Phi^+)$. *If* $\Gamma \subseteq \Phi_w$ *for some* $w \in W$ *and* Γ *is unipedal, then* $\bar{\Gamma} = \Phi_x \subseteq \Phi_w$ *for some* $x \in W$.

The previous theorem provides a new formula for the join (when defined) of elements of W in weak order. For $x, y, z \in W$, $x \vee y = z$ if and only if $\overline{\Phi_x \cup \Phi_y} = \Phi_z$. There is a similar (proven) formula for meet, which we do not give.

9.3 A Conjectural Normal Form for Elements of W

Our conjectural normal form for elements of W depends on the following conjecture of Dyer in [17].

Conjecture 9.3.1. Let $m \in \mathbb{N}$ and let $\Phi^+_{\leq m} = \{\alpha \in \Phi^+ | \ h^\infty(s_\alpha) \leq m\}$ be the subset of positive roots corresponding to $T_{\leq m}$. Then $\Phi^+_{\leq m}$ is balanced.

In fact, we will only need the weaker conjecture that $\Phi^+_{\leq m}$ is bipedal (see part 1 of Proposition 9.2.2). Then we have the following theorem.

Theorem 9.3.2. *Let $m \in \mathbb{N}$ such that $\Phi^+_{\leq m}$ is bipedal.*

1. *For any $x \in W$, there is a unique $x' \in W$ such that $N_m(x') = N_m(x)$, and any $y \in W$ with $N_m(y) \supseteq N_m(x)$ can be written $y = x'y''$ with $l(y) = l(x') + l(y'')$.*
2. *Any $1 \neq w \in W$ can be uniquely written as $w = w_1 \cdots w_n$ for certain $w_i \in W \setminus \{1\}$ with $i = 1, ..., n$ and $n \geq 1$ such that $l(w) = l(w_1) + \cdots + l(w_n)$ and $w_i = (w_i \cdots w_n)'$ for each i where $x \mapsto x'$ is as in 1.*

Proof. Let $x \in W$. Then Φ_x is biclosed and thus unipedal by part 3 of Proposition 9.2.2. By assumption, $\Phi^+_{\leq m}$ is bipedal and so $\Phi_x \cap \Phi^+_{\leq m}$ is unipedal by part 4 of 9.2.2. Therefore, according to Theorem 9.2.5, we see that $\overline{\Phi_x \cap \Phi^+_{\leq m}} = \Phi_{x'}$ for some $x' \in W$. If $y \in W$ with $N_m(y) \supseteq N_m(x)$, then $\Phi_x \cap \Phi^+_{\leq m} \subseteq \Phi_y \cap \Phi^+_{\leq m} \subseteq \Phi_y$. Therefore we get that

$$\Phi_{x'} = \overline{\Phi_x \cap \Phi^+_{\leq m}} \subseteq \overline{\Phi_y} = \Phi_y,$$

and so $y = x'y''$ with $l(y) = l(x') + l(y'')$ due to well known properties of the weak order on (W, S), which can be found in [2]. This proves 1, and 2 follows from 1 inductively. □

The next proposition shows that the previous theorem holds in many basic Coxeter systems.

Proposition 9.3.3. *Let (W, S) be a Coxeter system, and let $m \in \mathbb{N}$.*

1. *If (W,S) is finite or affine, then $\Phi^+_{\leq m}$ is bipedal.*

2. *If (W,S) is right-angled, that is $m(s,s') \in \{1,2,\infty\}$ for all $(s,s') \in S \times S$, then $\Phi^+_{\leq m}$ is bipedal.*

Proof. If (W,S) is finite, then $\Phi^+_{\leq m} = \Phi^+$ for all $m \in \mathbb{N}$, and so the result is trivial. Suppose that (W,S) is affine. Then there is a well-known description of the root system of (W,S) in terms of the root system, Ψ, of the associated finite Weyl group, (W°, S°). We remind the reader of this now (see [24]). We let U denote the span of Ψ, and let Δ be the simple roots for Ψ. We define $V := U + \mathbb{R}\delta$ to be a vector space with U as a subspace of codimension one. Then, we have $\Phi = \{\alpha + n\delta |\ \alpha \in \Psi;\ n \in \mathbb{Z}\}$,

$$\Phi^+ := \{\alpha + n\delta |\ \alpha \in \Psi^+;\ n \geq 0\} \cup \{-\alpha + n\delta |\ \alpha \in \Psi^+;\ n \geq 1\},$$

and $\Pi = \Delta \cup \{\delta - \widetilde{\alpha}\}$ is the set of simple roots for Φ where $\widetilde{\alpha}$ is the highest root for Ψ. It is well known that every reflection, $s_{\alpha+n\delta}$ $(\alpha \in \Psi)$, is contained in a unique infinite maximal dihedral reflection subgroup, namely the group generated by $\{s_\alpha, s_{\delta-\alpha}\}$. It follows from inspection in the dihedral case then that for $\alpha \in \Psi^+$ we get $h^\infty(s_{\alpha+n\delta}) = h^\infty(s_{-\alpha+(n+1)\delta}) = n$. Now, let $m \in \mathbb{N}$, and suppose $\alpha + e\delta \in \Phi_{\leq m}$. Let $s_{\alpha+e\delta} \in W'$ be a dihedral reflection subgroup with $\Pi_{W'} = \{\beta + n\delta, \gamma + k\delta\}$. Then, we have $\alpha + e\delta = a(\beta + n\delta) + b(\gamma + k\delta)$ with $a, b \in \mathbb{Z}_{>0}$. Therefore, $e = an + bk$, and so $an + bk \leq m$ with $a, b \in \mathbb{Z}_{>0}$. This implies that $n \leq m$ and $k \leq m$, and thus $h^\infty(\beta + n\delta) \leq n \leq m$ and $h^\infty(\gamma + k\delta) \leq k \leq m$. Hence $\Pi_{W'} \subset \Phi_{\leq m}$ as required.

To show 2, suppose that (W,S) is right-angled. We know that every dihedral reflection subgroup either has order 4 or ∞. Then for any $t \in T$ and $W' \in \mathscr{M}_t$

with $|W'| = 4$, we must have $t \in \chi(W')$ and so $h_{(W',\chi(W'))}(t) = 0$. Therefore, we must have

$$h^\infty(t) = \sum_{\substack{W' \in \mathscr{M}_t \\ |W'|=\infty}} h_{(W',\chi(W'))}(t) = \sum_{W' \in \mathscr{M}_t} h_{(W',\chi(W'))} = h(t).$$

Now suppose $\alpha \in \Phi^+_{\leq m}$ and $W' \in \mathscr{M}_{s_\alpha}$. Let $\{\beta, \gamma\} = \Pi_{W'}$. Then, by previous statement and the fact that $s_\beta \leq_\emptyset s_\alpha$ in Bruhat order, we get that

$$h^\infty(s_\beta) = h(s_\beta) \leq h(s_\alpha) = h^\infty(s_\alpha) \leq m,$$

and so $\beta \in \Phi^+_{\leq m}$, and similarly for γ as required. □

This normal form described by Theorem 9.3.2 could be called the *m-automatic normal form* for reasons we now describe. Suppose $m \in \mathbb{N}$ is fixed. Let $p : T \to \mathbb{N}$ be given by $p(t) = 1$ for all $t \in T$. Then $l_p = l$, and we can construct the m-canonical automata from Chapter 5, $\mathfrak{M}_m$. If we input a reduced expression for $w \in W$ in to the automata starting at $(m, N_m(1))$, we end on the node corresponding to $(m, N_m(w))$. According to the conjecture, there is a unique minimal element $x \in W$ with $(m, N_m(x)) = (m, N_m(w))$. This element is w_1. Then, we repeat this process with $w_1^{-1}w$ and so on until we are finished. Since this normal form comes from the m-canonical automata, the name is justified.

Remark 9.3.4. Let p be as in Chapter 3 with associated set $\mathcal{M}$. For any $\mathbf{m} \in \mathcal{M}$, we can define $\Phi^+_{\leq \mathbf{m}} := \{\alpha \in \Phi^+ | \, h_{W,p,\mathscr{M}_\infty}(s_\alpha) \leq \mathbf{m}\}$. Following Conjecture 9.3.1, it is reasonable to conjecture that $\Phi^+_{\leq \mathbf{m}}$ is balanced (or weaker, bipedal). We include the following example to show that this is in fact not the case in general.

Example 9.3.5. Let (W, S) be the affine Weyl group of type $\tilde{B}_2$. That is $S =$

$\{r, s, t\}$ with $m(r, s) = m(s, t) = 4$ and $m(r, t) = 2$. Due to well known facts, there are 8 elementary reflections. We list this set below.

$$T_0 = \{r, rsr, srs, s, sts, tst, t, rtstr\}.$$

Now, there are three conjugacy classes of reflections. Define $p : T \to \mathbb{N}^3$ by $p(u) = (1, 0, 0)$ if u is conjugate to r, $p(u) = (0, 1, 0)$ if u is conjugate to s, and $p(u) = (0, 0, 1)$ if u is conjugate to t; let $l_p : W \to \mathbb{N}^3$ and $h_{W,p,\mathscr{M}_\infty} : T \to \mathbb{N}^3$ be the corresponding length and infinite height functions.

Next, let $\Phi^+_{\leq(0,1,0)}$ be as defined in the previous remark. Define

$$W' := \langle rstsr, rsrtstrsr, srstsrs, s\rangle$$

so that $\chi(W') = \{rstsr, s\}$. Then the reflections of W' have the following infinite heights: $h_{W,p,\mathscr{M}_\infty}(s) = (0, 0, 0)$, $h_{W,p,\mathscr{M}_\infty}(rstsr) = (1, 0, 0)$, $h_{W,p,\mathscr{M}_\infty}(rsrtstrsr) = (0, 1, 0)$, and $h_{W,p,\mathscr{M}_\infty}(srstsrs) = (1, 0, 0)$. Then clearly, $\alpha_{rsrtstrsr} \in \Phi^+_{\leq(0,1,0)}$ and $rsrtstrsr \in W'$, but $\alpha_{rstsr} \in \Pi_{W'}$ and $\alpha_{rstsr} \notin \Phi^+_{\leq(0,1,0)}$. Therefore $\Phi^+_{(0,1,0)}$ is not bipedal. We note that (according to Proposition 9.3.3) this example does not provide a counterexample to Conjecture 9.3.1.

CHAPTER 10

HYPERBOLIC COXETER GROUPS

The standard classification of finite reflection groups (or finite Coxeter systems) and affine Coxeter systems can be found in [23] or [24]. Affine Coxeter systems are in some natural sense the smallest infinite Coxeter groups (e.g. they have only polynomial growth, see Chapter 12). In this chapter, we will consider the "next" class of infinite Coxeter systems known as hyperbolic Coxeter systems. There is a very well known, case-by-case, classification of hyperbolic Coxeter systems, which can be found in [23, chapter 6]. In this chapter, we discuss several characterizations of hyperbolic Coxeter systems by properties of their Coxeter graphs. According to [6], it is well known that any irreducible, infinite, non-affine Coxeter system contains a standard parabolic subsystem that is a hyperbolic Coxeter system; however, we could not find a proof in the literature, and the proof of the characterizations we include allows us to provide a case-free proof of this fact. For the remainder of this chapter, we will assume that S is a finite set.

10.1 Coxeter Graphs

Fix a Coxeter system, (W, S). We let $\Gamma := \Gamma_{(W,S)}$ be the associated Coxeter graph; that is, Γ is the undirected, labeled graph with vertex set S and edges (s_i, s_j) whenever $m(s_i, s_j) \geq 3$ and each edge labeled by the corresponding order

$m(s_i, s_j)$. To this graph, we associate the matrix of the bilinear form defined in section 2.2. That is, A is a symmetric matrix with $A_{i,j} := A_{s_i,s_j} = (\alpha_i, \alpha_j)$. We say that (W, S) is irreducible if Γ is connected. For $J \subseteq S$, let Γ_J be the full subgraph of Γ with vertices corresponding to J. We may abuse notation and say that $s \in S - J$ is connected to J if s is connected to Γ_J.

10.2 Symmetric Matrices and Sylvester's Law

For any real symmetric matrix over a finite dimensional vector space, we define the inertia of A to be the triple (a, b, c) where a represents the number of positive eigenvalues of A, b represents the number of negative eigenvalues of A, and c is the number of zero eigenvalues of A. We will abuse notation and say that a Coxeter system has inertia (a, b, c) if the corresponding matrix A has inertia (a, b, c). We say that an $n \times n$ symmetric matrix A is positive definite if it has inertia $(n, 0, 0)$. We say that a $n \times n$ symmetric matrix is positive semidefinite if it has inertia $(a, 0, c)$, i.e. it has only non-negative eigenvalues. If A is positive definite or positive semi-definite then we say A is of positive type. We state the following fact of real symmetric matrices (see for example [22]):

Proposition 10.2.1 (Sylvester's law of inertia)**.** *If A is an $n \times n$ real symmetric matrix with inertia (a, b, c), then A is congruent to the diagonal matrix $D = I_a \oplus -I_b \oplus \mathbf{0}_c$.*

10.3 Hyperbolic Coxeter Systems

Let (W, S) be an irreducible Coxeter system with associated graph Γ (so Γ is connected) and matrix A (representing the bilinear form as in 2.2). Suppose S is finite and $|S| = n$. Following [23], we know that W is finite if and only if A is positive definite, that is, if A has inertia $(n, 0, 0)$. Also, we know that W is affine if and only if the inertia is $(n-1, 0, 1)$. We define an irreducible Coxeter system to be **hyperbolic** if it has inertia $(n-1, 1, 0)$ and $(v, v) < 0$ for all $v \in C := \{\lambda \in V | \ (\lambda, \alpha_s) > 0 \ \forall s \in S\}$. Also, we have the following characterization of hyperbolic Coxeter systems:

Proposition 10.3.1. *[23, Prop. 6.8] Let (W, S) be an irreducible Coxeter system, with graph Γ and associated bilinear form $(-,-)$ with matrix A. Then (W, S) is hyperbolic if and only if the following two conditions are satisfied:*

1. *$(-,-)$ (or equivalently A) is non-degenerate, but not positive definite.*
2. *For each $s \in S$, the Coxeter graph Γ', obtained by removing s from Γ, is of positive type.*

10.4 Non-affine Coxeter Systems

At this point, we prove a quick technical lemma that will be needed in what is to follow. From this point on, we may use (W, S), W and $\Gamma = \Gamma_{(W,S)}$ interchangeably to represent the corresponding Coxeter system. There is no confusion in doing this as every finite rank (W, S) has a corresponding Coxeter graph and every simple finite, undirected, edge-labeled graph (labels in $\mathbb{N}_{\geq 3} \cup \{\infty\}$) gives rise to a finite rank Coxeter system. We say a graph Γ is of finite (resp. affine) type if the corresponding Coxeter system is finite (resp. affine).

Lemma 10.4.1. *Suppose that (W, S) is an infinite, irreducible, non-affine Coxeter system. Let $|S| = n$. Suppose that there exists $s \in S$ such that the standard parabolic subsystem $(W_{S \setminus \{s\}}, S \setminus \{s\})$ is a connected affine Coxeter system. Then the inertia of (W, S) is $(n-1, 1, 0)$.*

Proof. Let (W, S) be as in the statement. Thus, there exists $s \in S$ such that $S \setminus \{s\}$ is of connected affine type. By removing another simple reflection s' we must get a finite Coxeter system of rank $n-2$, and so (W, S) must have at least $n-2$ positive eigenvalues.

Now, we order S so that $S = \{s_1, ..., s_{n-1}, s_n\}$ where $s_n = s$. Then, we let A' be the $(n-1) \times (n-1)$ leading principal minor of the matrix A associated to (W, S) and Γ. Since A' is the matrix for $(W_{S\setminus\{s\}}, S \setminus \{s\})$ which is of affine type of rank $n-1$, it has inertia $(n-2, 0, 1)$ and there exists an isotropic root called δ. Additionally, we know that $\delta = \sum_{i=1}^{n-1} k_i \alpha_{s_i}$ with $k_i > 0$ for all i (c.f. [24]). By Proposition 10.2.1, we know that there exists an invertible matrix C such that $C^{tr} A' C = I_{n-2} \oplus \mathbf{0}_1$. We can lift this to a matrix $D = C \oplus I_1$ so that

$$B := D^{tr} A D = \begin{pmatrix} \begin{matrix} \\ & I_{n-2} & \\ \\ \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} a_1 \\ \vdots \\ a_{n-2} \end{matrix} \\ 0 \cdots 0 & 0 & c \\ a_1 \cdots a_{n-2} & c & 1 \end{pmatrix}$$

where $c = (\alpha_s, \delta)$ and $a_i \in \mathbb{R}$ for all i. Since (W, S) is irreducible, Γ is connected and so there exists some $i \neq n$ such that $(\alpha_{s_n}, \alpha_{s_i}) < 0$. Thus by the characterization of δ above we see that $(\alpha_{s_n}, \delta) \neq 0$, i.e. $c \neq 0$.

Next, we want to find the determinant of B. To do this, we expand the matrix along the $(n-1)$'st row noting that we have to multiply c by -1 due to its position in the matrix. We thus get that $\det(B) = -c \cdot \det(B')$ where

$$B' := \begin{pmatrix} I_{n-2} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ a_1 \cdots a_{n-2} & c \end{pmatrix}$$

Now, we can find the determinant of B' easily by expanding along the $(n-1)$'st column of B' (where we will multiply c by $+1$ due to its position in the matrix. This gives us that $\det(B') = c\det(I_{n-2})$. Putting the results together we see that

$$\det(B) = -c\det(B') = -c^2\det(I_{n-2}) = -c^2.$$

Since B has at least $n-2$ positive eigenvalues and it has negative determinant (due to the fact that $c \neq 0$), we know that exactly one of the remaining eigenvalues must be negative. Indeed, if both were negative or both were positive, the determinant of B would be positive. Thus B has inertia $(n-1, 1, 0)$. Using Proposition 10.2.1, we can conclude that A must have inertia $(n-1, 1, 0)$ as well. □

10.5 A Class of Coxeter Systems

We define a class of Coxeter systems as follows.

Definition 10.5.1. Let $\mathbf{H}$ be the class of all irreducible, finite rank, infinite, non-affine Coxeter systems, (W, S), with Coxeter graph Γ such that every proper subgraph $\Gamma' \subset \Gamma$ is the disjoint union of graphs that are Coxeter graphs of only finite or affine type.

Lemma 10.5.2. *Every hyperbolic Coxeter system is in* $\mathbf{H}$.

Proof. By Proposition 10.3.1 part (2) we see that any hyperbolic Coxeter system must be in $\mathbf{H}$. In particular, the hyperbolic Coxeter systems are precisely those in $\mathbf{H}$ with inertia $(n-1, 1, 0)$. □

10.6 A Smaller Class of Coxeter Systems

We now define a certain subclass of $\mathbf{H}$ that will be interesting to study.

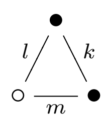
Definition 10.6.1. Let $\mathbf{h} \subseteq \mathbf{H}$ be the class of all Coxeter systems, (W, S), in $\mathbf{H}$ that have Γ satisfying exactly one of the following (it will be clear that a graph cannot satisfy both):

(**h1**) There exists $s \in S$ such that $\Gamma - \{s\}$ is a disjoint union of Coxeter graphs of finite type.

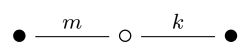
(**h2**) $\Gamma - \{s\}$ is a connected Coxeter graph of affine type for all $s \in S$.

Example 10.6.2. Notice that any infinite rank 3 Coxeter system can be described by one of the two following graphs (note: we color a vertex $\circ$ if we intend to refer to it below):

(a)



(b)



with $l \geq m \geq k \geq 3$. For arbitrary m and k, if we remove vertex $\circ$ from graph (b) we will be left with the Coxeter graph for a dihedral Coxeter system of order 4. Thus, graph (b) is clearly of type (**h1**) (provided it is infinite and non-affine).

For graph (a), if $k \neq \infty$, then removing vertex $\circ$ leaves us with the Coxeter graph for a finite dihedral Coxeter system, and thus (a) is of type (**h1**) (provided it is not affine). If $k = \infty$ then we necessarily have $m = \infty$ and $l = \infty$. Therefore, removing any vertex from (a) gives us the graph $\bullet \overset{\infty}{\text{---}} \bullet$, which is clearly of affine type. Thus any infinite, non-affine irreducible rank 3 Coxeter system is in $\mathbf{h}$.

Lemma 10.6.3. *Any Coxeter system in* $\mathbf{h}$ *is hyperbolic.*

Proof. Let (W, S) be in $\mathbf{h} \subseteq \mathbf{H}$ and let $|S| = n$, i.e. the rank of (W, S) is n. Let Γ be the Coxeter graph of (W, S) and let A be the matrix associated to $(-,-)$ (or similarly to Γ).

Case 1: (W,S) is of type (**h1**). Then, since there exists $s \in S$ such that $(W_{S-\{s\}}, S-\{s\})$ is of finite type and so the associated bilinear form and matrix A' must have inertia $(n-1,0,0)$. Thus A must have at least $n-1$ positive eigenvalues. If A has inertia $(n,0,0)$ (resp. $(n-1,0,1)$) then (W,S) would be of finite type (resp. affine type) and thus $(W,S) \notin \mathbf{H}$, which is a contradiction (since $\mathbf{h} \subseteq \mathbf{H}$ by definition). Therefore A must have inertia $(n-1,1,0)$ forcing (W,S) to be hyperbolic by Proposition 10.3.1.

Case 2: (W,S) is of type (**h2**). Then since $S-\{s\}$ is of connected affine type Lemma 10.4.1 implies that the inertia of (W,S) is $(n-1,1,0)$. In particular $(-,-)$ is non-degenerate. Furthermore, since for each $r \in S$ we have $\Gamma - \{r\}$ is of positive type, Proposition 10.3.1 implies that (W,S) is hyperbolic.

□

10.7 Parabolic Subsystems of Non-affine Coxeter Systems

Lemma 10.7.1. *Any infinite, irreducible, finite rank, non-affine Coxeter system (W,S) contains a standard parabolic subsystem in the class $\mathbf{H}$.*

Proof. Let (W',S') be a minimal rank, irreducible, infinite, non-affine parabolic subsystem of (W,S), which must exist. Then since (W',S') is minimal rank, if we remove any vertex s then $(W_{S'\setminus\{s\}}, S'\setminus\{s\})$ cannot contain an infinite, non-affine parabolic subsystem. Thus, $(W_{S'\setminus\{s\}}, S'\setminus\{s\})$ must be of finite or affine type. □

10.8 Characterizations of Hyperbolic Coxeter Systems

Lemma 10.8.1. *Let (W,S) be in the class $\mathbf{H}$. Then (W,S) is in the class $\mathbf{h}$.*

Proof. Suppose that (W, S) is not of type (**h1**) and let $\Gamma := \Gamma_S$. Then because (W, S) is in class **H**, we know that $(W_{S\setminus\{s\}}, S \setminus \{s\})$ is of affine type for every $s \in S$. Suppose that $s \in S$ and $\Gamma_{S\setminus\{s\}}$ is not connected. Let $\Gamma_0, \Gamma_1, ..., \Gamma_n$ be the connected components (so $n \geq 1$). Now, since $\Gamma_{S\setminus\{s\}}$ is of affine type we can assume without loss of generality that Γ_0 is of affine type. Then $\Gamma' = \Gamma_0 \cup \{s\}$ is irreducible (since (W, S) is irreducible) and is properly contained in $\Gamma_0 \cup \{s\} \cup \{s_1\}$ where $s_1 \in \Gamma_1$. Thus Γ' is properly contained in Γ. However, Γ' cannot be of finite or affine type or else its subgraph Γ_0 would be finite, contradicting that Γ_0 is affine. Therefore if $\Gamma_{S\setminus\{s\}}$ is not connected then Γ properly contains Γ' which is infinite and non-affine, contradicting the assumption that (W, S) is in **H**. Thus, we have shown that if (W, S) is not of type (**h1**), then for every $s \in S$, $\Gamma \setminus \{s\}$ is of connected affine type, i.e. (W, S) is of type (**h2**). □

Theorem 10.8.2. *The following hold:*

1. *The classes* **h** *and* **H** *both coincide with the class of hyperbolic Coxeter systems.*

2. *Any finite rank, irreducible, infinite, non-affine Coxeter system contains a standard parabolic subsystem of hyperbolic type.*

Proof. By Lemma 10.5.2, we know that all hyperbolic Coxeter systems are contained in **H**. Then, by Lemma 10.8.1 we know that **H** is contained in **h** (so clearly **h** and **H** are equal). Finally, by Lemma 10.6.3 we know that every element of **h** is a hyperbolic Coxeter system. Thus we have equality throughout and 1 follows.

For 2, we use part 1 of the theorem just proved along with Lemma 10.7.1. □

CHAPTER 11

LARGE REFLECTION SUBGROUPS

By a universal Coxeter system, we mean one with no braid relations (the underlying group is a free product of cyclic groups of order 2). The main result of this chapter is that all irreducible, non-affine Coxeter systems have universal reflection subgroups of arbitrarily large rank. To prove this, we use the imaginary cone, which was introduced by Dyer ([14]) in order to prove a conjectured characterization of coverings in dominance order (see Remark 2.5.3). Dyer raised the question of whether the imaginary cone of an infinite, irreducible, non-affine Coxeter system can be approximated by imaginary cones of universal reflection subgroups. We show this is the case for hyperbolic Coxeter groups, and then our main result follows from Chapter 10. Many of the auxiliary results in this chapter can be found in [15]. Since we will define a few different cones, we need some terminology regarding convexity and polyhedral cones. For a reference on such terms, consult [1].

11.1 Convex Sets

In this section, we describe our terminology involved with convex sets. Let V be a real vector space.

Definition 11.1.1. We say that x is a convex combination of $x_1, ...x_n \in V$ if there exists $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that

1. $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$

2. $\lambda_1 + \cdots + \lambda_n = 1$

3. $\lambda_i \geq 0$ for all i.

For any set M we call the set of all convex combinations of elements of M the convex hull of M and we denote it by $\text{conv}(M)$.

For any two elements $x, y \in V$, we denote

$$[x, y] := \{\lambda x + (1 - \lambda)y |\ 0 \leq \lambda \leq 1\}.$$

We say a set $C \subset V$ is *convex* if, for all $x, y \in C$ $(x \neq y)$, $[x, y] \subset C$.

11.2 A Metric on Convex Sets

Assume that V is finite dimensional, so that $V \cong \mathbb{R}^n$. Then V has the usual distance function as a metric $d : V \times V \to \mathbb{R}$. Let $Y \subseteq V$ and $x \in V$. Define

$$d(x, Y) = \inf_{y \in Y} d(x, y) \tag{11.2.1}$$

Then for any $X \subseteq V$ and $Y \subseteq V$, we define

$$d(X, Y) = \sup_{x \in X} d(x, Y) \tag{11.2.2}$$

Definition 11.2.1. Let $K := \{X \subseteq V |\ X \text{ is compact}\}$.

The following proposition can be found in [25, §21 VII].

Proposition 11.2.2. *Consider the distance function defined in* (11.2.2). *We create a new distance function on* K *given by* $d_K(X, Y) = \max\{d(X, Y), d(Y, X)\}$. *Then* d_K *defines a metric on* K.

The metric defined above is known as the Hausdorff metric (or Hausdorff distance).

11.3 Approximation of a Sphere by Polytopes

Now, suppose that S^d is the d-sphere sitting inside of $\mathbb{R}^{d+1}$. We want to show that S^d can be approximated by the boundary of a convex hull of a finite number of points. For any set $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A. To make the previous statement more concrete, we have the following lemma.

Lemma 11.3.1. *There exists a sequence of compact, convex sets* P_m *with* $P_m =$ *conv*$(\{x_1, ..., x_m\})$ *where* $\{x_1, ..., x_m\} \subset S^d$ *and* $m < \infty$ *such that* $d_K(\partial P_m, S^d) \to 0$ *as* $m \to \infty$, *i.e.* $\partial P_m \to S^d$ *in the metric space* K.

Proof. Let $\epsilon > 0$ be given. Now, for any point $z \in \mathbb{R}^{d+1}$, define $\mathscr{B}_a(z)$ to be the open ball of radius a centered at z. Then we define

$$\mathcal{U} := \{\mathscr{B}_{\frac{\epsilon}{2}}(z) |\ z \in S^d\}.$$

Clearly $S^d \subset \bigcup_{U \in \mathcal{U}} U$. Since S^d is compact, we can find a finite open subcover of $\mathcal{U}$ call it $\mathcal{U}' := \{U_1, ..., U_m\}$ (where $m = m_\epsilon$ is dependent on ϵ).

Now, each U_i is a ball of radius $\epsilon/2$ centered at $x_i \in S^d$. Thus, we have a finite set of points $\{x_1, ..., x_m\} \subset S^d$. Let $P_m := \text{conv}(\{x_1, ..., x_m\})$. Let ∂P_m denote the boundary of P_m.

Suppose $y \in \partial P_m$. Now, there exists a minimal set $I := \{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_m\}$ such that $y \in \text{conv}(\{x_{i_1}, ..., x_{i_k}\} = \text{conv}(I) \subset \partial P_m$. Then, we can choose a supporting affine hyperplane H of P_m such that $I \subset P_m \cap H$. Let H^+ and H^- represent the positive and negative half-spaces associated to H. Suppose, without loss of generality, that $P_m \subset H^-$. Let $H' := \text{int}(H^+)$ and let u be an outer unit normal vector of H (u points into H^+). By assumption, $P_m \cap H' = \emptyset$ and so $\partial P_m \cap H' = \emptyset$ and $\{x_1, ..., x_m\} \cap H' = \emptyset$.

Next, for the sake of contradiction, we suppose that $d(y, S^d) \geq \epsilon$. Then let $L = H + \frac{\epsilon}{2}u$ be the affine hyperplane parallel to H with $d(H, L) = \frac{\epsilon}{2}$ and $L \subset H'$. It follows that $S^d \cap L^+ \neq \emptyset$, where L^+ is the positive half-space associated to L. Indeed, if $S^d \cap L^+ = \emptyset$, then $S^d \subseteq L^-$ and so $d(P_m \cap H, S^d) \leq d(H, L)$. This would imply that

$$d(y, S^d) \leq \sup_{x \in P_m \cap H} d(x, S^d) = d(P_m \cap H, S^d) \leq d(H, L) = \frac{\epsilon}{2} < \epsilon,$$

which is not true by assumption. Thus, we let $z \in S^d \cap L^+$. Then $\mathscr{B}_{\frac{\epsilon}{2}}(z) \subset H'$ by construction. However, $z \in S^d$ so $z \in U_i$ for some $U_i \in \mathcal{U}'$ so $x_i \in \mathscr{B}_{\frac{\epsilon}{2}}(z) \subset H'$ which is a contradiction. Therefore, $d(y, S^d) < \epsilon$.

Since y is arbitrary, we can conclude that $d(\partial P_m, S^d) < \epsilon$. Now, suppose $y \in S^d$, then $y \in U_i$ for some i. Thus,

$$d(y, \partial P_m) = \inf_{z \in \partial P_m} d(y, z) \leq d(y, x_i) < \frac{\epsilon}{2}.$$

Again, since y is arbitrary, we can conclude that $d(S^d, \partial P_m) < \frac{\epsilon}{2}$. Thus we have determined that $d_K(\partial P_m, S^d) = \max\{d(S^d, \partial P_m), d(\partial P_m, S^d)\} < \max\{\frac{\epsilon}{2}, \epsilon\} = \epsilon$ as desired. $\square$

11.4 Cones in Coxeter Systems

Recall, for any reflection subgroup W', we let $\chi(W')$ be the set of canonical generators of W' as a Coxeter system. Then, according to section 2.3, we have a corresponding set of roots, positive roots, and simple roots for the Coxeter system $(W', \chi(W'))$ given by

$$\Phi_{W'} = \{\alpha \in \Phi |\ s_\alpha \in W'\}, \quad \Phi^+_{W'} := \Phi_{W'} \cap \Phi^+, \quad \Pi_{W'} = \{\alpha \in \Phi^+ |\ s_\alpha \in \chi(W')\}$$

respectively.

We now introduce several cones that appear inside of the vector space V introduced in section 2.2. For any W' a reflection subgroup of W, we define the fundamental chamber for W' on the vector space V by

$$\mathscr{C}_{W'} = \{v \in V |\ (v, \alpha) \geq 0 \text{ for all } \alpha \in \Pi_{W'}\}.$$

Then we denote the Tits cone of W' by $\mathscr{X}_{W'}$ where this is defined by

$$\mathscr{X}_{W'} = \bigcup_{w \in W'} w(\mathscr{C}_{W'}).$$

In addition to these we define the following:

$$\mathscr{K}_{W'} := \mathbb{R}_{\geq 0}\Pi_{W'} \cap -\mathscr{C}_{W'}, \qquad \mathscr{Z}_{W'} := \bigcup_{w \in W'} w(\mathscr{K}_{W'})$$

where $\mathscr{Z}_{W'}$ is known as the imaginary cone of W' on V (we only define it if S is finite). The definition of the imaginary cone here is taken from [15], which models it loosely on one characterization of imaginary cones of Kac-Moody Lie algebras

([24]). The results we use are analogs for general W of results proved by Kac for imaginary cones of Kac-Moody Lie algebras (for which, however, the proofs use imaginary roots, which have no counterpart here). We will refer to $\mathscr{C}_W, \mathscr{X}_W, \mathscr{K}_W$ and $\mathscr{Z}_W$ simply by $\mathscr{C}, \mathscr{X}, \mathscr{K}$ and $\mathscr{Z}$.

11.5 Topology on Rays in Cones

By a *ray* through v in V we mean a set $\{\lambda v | \ \lambda \in \mathbb{R}_{\geq 0}\}$. We let $\mathscr{R}$ denote the set of rays of V. We can give $\mathscr{R}$ a topology and a metric in the following way. Let B be compact convex body containing 0 in V, and let ∂B denote the boundary of B. Define the map $\varphi : \mathscr{R} \to \partial B$ by $\varphi(x) = x \cap \partial B$. It is clear that φ is a bijection. We thus declare φ to be a homeomorphism between $\mathscr{R}$ and ∂B, and this gives $\mathscr{R}$ a topology and a metric corresponding to ∂B in V, with the topology independent of B. We also introduce the following subsets of $\mathscr{R}$. Let $\mathscr{R}_+ := \{\mathbb{R}_{\geq 0}\alpha | \ \alpha \in \Phi^+\}$ and let $\mathscr{R}_0 := \overline{\mathscr{R}_+} \setminus \mathscr{R}_+$. We say a ray, $\mathbb{R}_{\geq 0}\alpha$, is positive if $(\alpha, \alpha) > 0$ and isotropic if $(\alpha, \alpha) = 0$. We recall some facts about these sets (both proven in [15]).

Proposition 11.5.1. *1. $\mathscr{R}_+$ consists of positive rays and is discrete in the subspace topology.*

2. $\mathscr{R}_0$ consists of isotropic rays and is closed in $\mathscr{R}$.

3. $\mathscr{R}_0$ is the set of limit rays of $\mathscr{R}_+$.

In addition, we have the following theorem.

Theorem 11.5.2. *The cone $\overline{\mathscr{Z}}$ is the convex closure of $\bigcup_{r \in \mathscr{R}_0} r \cup \{0\}$.*

This theorem has the following immediate corollary.

Corollary 11.5.3. *If W' is a finitely generated reflection subgroup of W, then $\overline{\mathscr{Z}_{W'}} \subseteq \overline{\mathscr{Z}_W}$.*

Remark 11.5.4. A much stronger fact, proven in [15], which we will not use, is that in the situation of the corollary above we actually have $\mathscr{Z}_{W'} \subseteq \mathscr{Z}_W$.

11.6 Cones in Hyperbolic Coxeter Systems

Recall that we say (W, S) is a hyperbolic Coxeter system if the bilinear form $(-,-)$ on V has inertia $(n-1, 1, 0)$ and every proper standard parabolic Coxeter system is of finite or affine type (see Chapter 10).

Suppose we fix a hyperbolic Coxeter system (W, S). We know that V has a basis $x_0, ..., x_n$ such that $(x_i, x_j) = \delta_{i,j}$, $(x_i, x_i) = 1$ for all $i \in \{1, ..., n\}$, and $(x_0, x_0) = -1$. Then, according to [15], we have (after replacing x_0 by $-x_0$ if necessary) that

$$\overline{\mathscr{Z}} = \overline{\mathscr{L}}$$

where $\overline{\mathscr{L}} = \{\sum_{i=0}^{n} \lambda_i x_i |\ \lambda_0 \geq \sqrt{\sum_{i=1}^{n} \lambda_i^2}\}$.

Now, consider the setup as in section 11.5. Let $\mathscr{R}_\Pi \subset \mathscr{R}$ be the set of rays spanned by non-zero elements of $\mathbb{R}_{\geq 0}\Pi$. Since (W, S) is hyperbolic, the bilinear form is non-degenerate, and thus the interior of $\mathscr{C}$ is non-empty (see [15]). Take $\rho \in \mathrm{int}(\mathscr{C})$ so that $(\rho, \alpha) > 0$ for all $\alpha \in \Pi$. Therefore, we may define a map

$$\tau : \mathbb{R}_{\geq 0}\Pi \setminus \{0\} \to H := \{v \in V |\ (v, \rho) = 1\}$$

given by $\tau(v) = v/(\rho, v)$. Then the image of τ is the set $P := \{v \in V |\ (\rho, v) = 1\}$, which is a convex polytope in the affine hyperplane H with points $(\rho, \alpha)^{-1}\alpha$ for $\alpha \in \Pi$ as vertices. From τ, we introduce a map on $\mathscr{R}$ as follows. For $r \in \mathscr{R}$, we let $\hat{\tau}(r) = \tau(x)$ for all $x \in r \setminus \{0\}$ (note $\hat{\tau}$ is independent of choice of $x \in r \setminus \{0\}$). With terminology from section 11.5 we may take B so that $P \subset \partial B$.

Now, we can see that $\overline{\mathscr{Z}} \cap H$ is a disk. Also, by Theorem 11.5.2 we see that $\overline{\mathscr{Z}} \cap H = \operatorname{conv}\left(\bigcup_{r \in \mathscr{R}_0} r\right) \cap H$.

Lemma 11.6.1. *$\mathscr{R}_0$ is the set of rays in the boundary of $\overline{\mathscr{Z}}$.*

Proof. First, a ray $r \in \mathscr{R}_0$ is isotropic by Proposition 11.5.1, and thus it is in $\pm\partial(\overline{\mathscr{L}})) = \pm\partial(\overline{\mathscr{Z}})$ due to the description of $\mathscr{L}$ above. However, r must also be in $\overline{\mathscr{Z}}$, so $r \in \partial(\overline{\mathscr{Z}})$.

For the reverse implication, the statements above imply it is enough to show that the result is true inside of H, that is the convex hull of limit points of (images of) roots in H includes the entire boundary sphere of $\left(\overline{\mathscr{Z}} = \overline{\mathscr{L}}\right) \cap H$. We let the boundary of $\overline{\mathscr{Z}} \cap H$ be denoted by $\mathcal{S} := \partial\hat{\tau}(\overline{\mathscr{Z}})$, and we let $A := \hat{\tau}\left(\operatorname{conv}\left(\bigcup_{r \in \mathscr{R}_0} r\right)\right)$. Then A is the convex hull of the points $\bigcup_{r \in \mathscr{R}_0} \hat{\tau}(r)$. But then the statement is clear since $\mathcal{S}$ is a sphere and is contained in A since $\overline{\mathscr{Z}} \cap H = \operatorname{conv}\left(\bigcup_{r \in \mathscr{R}_0} r\right) \cap H$. So every point in $\mathcal{S}$ must also be in A or else we would not have that the boundary of A is a sphere (which is true since (W, S) is hyperbolic). $\square$

11.7 Universal Coxeter Systems

Suppose we have a hyperbolic Coxeter system (W, S) as in section 11.6. Then with the same setup there, we know that $\overline{\mathscr{Z}} = \overline{\mathscr{L}}$ and intersecting with H we get that $\mathcal{S} = \partial\hat{\tau}(\overline{\mathscr{Z}})$ is a sphere in H.

Lemma 11.7.1. *Suppose that $x, y \in \mathcal{S}$. Then there exist $\alpha_x, \alpha_y \in \Phi^+$ such that $\tau(\alpha_x), \tau(\alpha_y) \in \hat{\tau}\left(\mathscr{R}_+\right)$ are sufficiently close to x and y respectively; furthermore, α_x and α_y satisfy $(\alpha_x, \alpha_y) < -1$. In particular, if $S' := \{s_{\alpha_x}, s_{\alpha_y}\}$ then (W', S') is an indefinite infinite dihedral reflection subgroup of (W, S) and $\chi(W') = S'$.*

Proof. Since $x, y \in \mathcal{S}$ then x and y are limit points of $\hat{\tau}(\mathscr{R}_+)$ and therefore we can find (images of) positive roots arbitrarily close to x and y. Thus, we pick positive roots $\alpha_x \in \Phi^+$ and $\alpha_y \in \Phi^+$ such that $\tau(\alpha_x)$ and $\tau(\alpha_y)$ are close enough to x and y respectively to ensure that $\{a\tau(\alpha_x) + b\tau(\alpha_y)| \ a, b \in \mathbb{R}_{>0}\} \cap (\mathscr{Z} \cap H) \neq \emptyset$. Again, this is possible since $\mathcal{S}$ is a sphere and x and y are two points ($x \neq y$) on the sphere and so the relative interior of the line connecting x and y must be included in the relative interior of the disk bounded by the sphere which is $\mathscr{Z} \cap H$. Now, we pick a point in the interior of the imaginary cone, $v = a\tau(\alpha_x) + b\tau(\alpha_y) \in \text{int}(\mathscr{Z} \cap H) \subset \text{int}(\mathscr{Z})$ with $a, b \in \mathbb{R}_{>0}$. Then $(v, v) < 0$ necessarily and so $(-,-)$ restricted to $\text{Span}\{\alpha_x, \alpha_y\}$ cannot be of finite type or affine type. Thus, (W', S') cannot be a finite dihedral subgroup. Therefore, we see that $|(\alpha_x, \alpha_y)| > 1$. Now if $(\alpha_x, \alpha_y) > 1$ then by [4, Proposition 2.2] (and without loss of generality) we must have α_x dominates α_y. However, this is impossible since, by investigation of dominance in infinite dihedral groups ([14]), we know that α_x can dominate α_y only if $[\alpha_x, \alpha_y] \cap \mathscr{Z}_{W'} = \emptyset$, which is not the case here. Thus, $(\alpha_x, \alpha_y) < -1$ and so $S' = \{s_{\alpha_x}, s_{\alpha_y}\}$ is the canonical set of generators for the infinite dihedral group generated by S', (W', S'). □

Now, by Lemma 11.3.1 recall that for arbitrary m we can find m points $\{x_1, ..., x_m\} \subset \mathcal{S}$ such that $P_m := \text{conv}\{x_1, ..., x_m\}$ along with the property that $d_K(P_m, \mathcal{S}) \to 0$ as $m \to \infty$. Then, by Lemma 11.7.1 above, since m is finite, we can find $\alpha_{x_i} \in \Phi^+$ with $\tau(\alpha_{x_i})$ sufficiently close to x_i such that $S'_{i,j} := \{s_{\alpha_{x_i}}, s_{\alpha_{x_j}}\}$ is the set of canonical generators for the (infinite) dihedral group it generates, $(W'_{i,j}, S'_{i,j})$. Now, let $S' := \{s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}}\}$ and W' be the reflection subgroup generated by S'.

Proposition 11.7.2. *Then (W', S') is a universal Coxeter system with $\chi(W') =$*

S'.

Proof. Consider the Coxeter system (W', S'). Then by [19, Proposition 3.5], $S' = \chi(W')$ if and only if $\{s, s'\} = \chi(\langle s, s'\rangle)$ for all $s, s' \in S'$ and this follows from Lemma 11.7.1. Finally, since $m(s, s') = \infty$ for all $s \neq s' \in S'$ then (W', S') is a universal Coxeter system. $\square$

11.8 Large Reflection Subgroups

With the terminology as in section 11.6, we have H containing P. Thus, we can view $\hat{\tau}(\overline{\mathscr{Z}})$ as a subset of P. Additionally, we can view any subset of $\mathscr{R}_\Pi$ as a subset of P. Thus, we will abuse notation by defining d_K on any compact subset of $\mathscr{R}_\Pi$ in the following way: for $X, Y \subset \mathscr{R}_\Pi$, $d_K(X, Y) = d_K(\hat{\tau}(X), \hat{\tau}(Y))$. We put the previous results together to obtain the following result.

Proposition 11.8.1. *Suppose (W, S) is hyperbolic. There exists a sequence of finite rank, universal reflection subgroups, (W'_m, S'_m), of W with $d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) \to 0$ as $m \to \infty$ and $d_K(\overline{\mathscr{Z}}_{W'_m}, \overline{\mathscr{Z}}) \to 0$ as $m \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Then we can pick m large enough so that $d_K(P_m, \mathcal{S}) < \epsilon/2$ by Lemma 11.3.1 where $P_m = \text{conv}\{x_1, ..., x_m\}$ for some $\{x_1, ..., x_m\} \subset \mathcal{S}$. Thus, $d_K(P_m, \overline{\mathscr{Z}} \cap H) < \epsilon/2$. According to Lemma 11.7.2 and the discussion before it, we can choose $\{\alpha_{x_1}, ..., \alpha_{x_m}\} \subset \Phi^+$ so that $\tau(\alpha_{x_i})$ is sufficiently close to x_i (in particular so that $d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/2$) with the property that (W'_m, S'_m) is a universal Coxeter system where $S'_m := \{s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}}\}$ and $W'_m = \langle S'_m\rangle$. Now, let $Q_m = \text{conv}\{\tau(\alpha_{x_1}), ..., \tau(\alpha_{x_m})\}$. Then $Q_m = \mathbb{R}_{\geq 0}\Pi_{W'_m} \cap H$ and $d_K(Q_m, P_m) < \epsilon/2$

since $d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/2$ (by assumption above). Therefore, we see that

$$\begin{aligned} d_K(\mathbb{R}_{\geq 0}\Pi_{W'}, \overline{\mathscr{Z}}) &= d_K(Q_m, \overline{\mathscr{Z}} \cap H) \\ &\leq d_K(Q_m, P_m) + d_K(P_m, \overline{\mathscr{Z}} \cap H) \\ &= d_K(Q_m, P_m) + d_K(P_m, \mathcal{S}) < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

as desired. It remains to show that we can take $d_K(\overline{\mathscr{Z}_{W'_m}}, \overline{\mathscr{Z}}) \to 0$ also.

We temporarily fix $i \neq j$. Then since $(\alpha_{x_i}, \alpha_{x_j}) < -1$, we have

$$\begin{aligned} \overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} &= \{z \in \mathbb{R}_{\geq 0}\alpha_{x_i} + R_{\geq 0}\alpha_{x_j} |\ (z, z) \leq 0\} \\ &= \mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i} \end{aligned}$$

(which is a two-dimensional cone) for some isotropic linearly independent vectors β_{x_i,x_j} and β_{x_j,x_i}. We choose our previous notation so that $\beta_{x_i,x_j} \in \mathbb{R}_{\geq 0}\alpha_{x_i} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}$ and $\beta_{x_j,x_i} \in \mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\alpha_{x_j}$.

Then, from [14], since $\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle$ is dihedral, we have

$$\overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} = \overline{\mathscr{Z}} \cap \left(\mathbb{R}_{\geq 0}\alpha_{x_i} + \mathbb{R}_{\geq 0}\alpha_{x_j}\right).$$

Thus, $\tau(\beta_{x_i,x_j})$ and $\tau(\beta_{x_j,x_i})$ are the points given by the intersection

$$[\tau(\alpha_{x_i}), \tau(\alpha_{x_j})] \cap \mathcal{S} = \{\tau(\beta_{x_i,x_j}), \tau(\beta_{x_j,x_i})\}.$$

This implies that we can choose α_{x_i} and α_{x_j} so that all of the following hold:

$$\begin{aligned} &d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/4 \qquad && d_K(\tau(\alpha_{x_j}), x_j) < \epsilon/4 \\ &d_K(\tau(\beta_{x_i,x_j}), x_i) < \epsilon/4 \qquad && d_K(\tau(\beta_{x_j,x_i}), x_j) < \epsilon/4. \end{aligned} \tag{11.8.1}$$

Next, by Corollary 11.5.3, we know that

$$\overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} \subseteq \overline{\mathscr{Z}}_{W'_m} \subseteq \overline{\mathscr{Z}}$$

for all pairs $i \neq j$. Each of these are convex cones, and so we also have

$$\sum_{i \neq j} \mathbb{R}_{\geq 0} \left(\mathbb{R}_{\geq 0} \beta_{x_i, x_j} + \mathbb{R}_{\geq 0} \beta_{x_j, x_i} \right) \subseteq \overline{\mathscr{Z}}_{W'_m} \subseteq \overline{\mathscr{Z}}. \tag{11.8.2}$$

Now, let $\epsilon > 0$. We choose m large enough and $\alpha_{x_1}, ..., \alpha_{x_m}$ sufficiently close to $x_1, ..., x_m$ respectively so that (11.8.1) holds and so that $d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) < \epsilon/2$ where $W'_m = \langle s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}} \rangle$ (which is possible by the first part of the proposition). Then (11.8.1) implies that $d_K(\alpha_{x_i}, \beta_{x_i, x_j}) < \epsilon/2$ for all $j \neq i$ so that

$$d_K([\tau(\beta_{x_i, x_j}), \tau(\beta_{x_j, x_i})], [\tau(\alpha_{x_i}), \tau(\alpha_{x_j})]) < \epsilon/2$$

and thus

$$d_K \left(\sum_{i \neq j} \left(\mathbb{R}_{\geq 0} \beta_{x_i, x_j} + \mathbb{R}_{\geq 0} \beta_{x_j, x_i} \right), \sum_{k=1}^{m} \mathbb{R}_{\geq 0} \alpha_{x_k} \right) < \frac{\epsilon}{2}.$$

Together, since $\sum_{k=1}^{m} \mathbb{R}_{\geq 0} \alpha_{x_k} = \mathbb{R}_{\geq 0} \Pi_{W'_m}$, we get that

$$d_K \left(\sum_{i \neq j} \left(\mathbb{R}_{\geq 0} \beta_{x_i, x_j} + \mathbb{R}_{\geq 0} \beta_{x_j, x_i} \right), \overline{\mathscr{Z}} \right) \leq$$
$$d_K \left(\sum_{i \neq j} \left(\mathbb{R}_{\geq 0} \beta_{x_i, x_j} + \mathbb{R}_{\geq 0} \beta_{x_j, x_i} \right), \sum_{k=1}^{m} \mathbb{R}_{\geq 0} \alpha_{x_k} \right) + d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then, equation (11.8.2) implies that $d_K(\overline{\mathscr{Z}}_{W'_m}, \overline{\mathscr{Z}}) < \epsilon$ as well, finishing the proof.

□

The next theorem follows from the previous propositions and Chapter 10.

Theorem 11.8.2. *Let (W, S) be a finite rank, irreducible, infinite, non-affine Coxeter system. Then, for any $m \in \mathbb{N}$, W contains a reflection subgroup W' with $|\chi(W')| = m$ such that $(W', \chi(W'))$ is a universal Coxeter system.*

Proof. Let $m \in \mathbb{N}$. By Theorem 10.8.2, (W, S) contains a hyperbolic parabolic subsystem (W_J, J). By Proposition 11.7.2, we have that (W_J, J) contains a universal Coxeter system (W', S') of rank m and thus (W, S) also contains (W', S') proving the theorem. □

CHAPTER 12

EXPONENTIAL GROWTH IN COXETER GROUPS

For any finitely generated group, we can define the growth function of the group by determining the number of elements of the group of all lengths (with respect to the generating set). In [6], de la Harpe demonstrates that irreducible, infinite, non-affine Coxeter systems have what is known as exponential growth by showing that any such Coxeter system must contain a free non-abelian subgroup. Also, in [27], Margulis and Vinberg prove the stronger result that such a W must contain a finite index subgroup which surjects onto a non-abelian free group. Finite and affine Coxeter systems are known to have polynomial growth.

Even more recently, in [29], Viswanath uses an alternate method to show that any irreducible, infinite, non-affine *simply-laced* Coxeter system has exponential growth. He also obtains the stronger result that for such a system (W, S) and any $J \subsetneq S$ the quotient W/W_J also has exponential growth. His method involves finding a particular universal rank 3 reflection subgroup inside of (W, S). In [29, Remark 3], the author notes that it would be interesting to know if this result holds in the non-simply laced case as well. We intend to answer this question using our results from the previous two chapters.

12.1 Growth Types

To begin with, we introduce the basic notions of growth types in finitely generated groups. Let (W, S) be a finitely generated Coxeter system. We follow [7] or [29] for general terminology and results.

Definition 12.1.1. Let $\mathbf{a} := (a_k)_{k\geq 0}$ be a sequence of non-decreasing natural numbers. We define the *exponential growth rate* of the sequence to be $\omega(\mathbf{a}) := \limsup_{k\to\infty} a_k^{1/k}$.

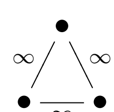
For each $k \in \mathbb{N}$, we define a subset $W_{\leq k} := \{w \in W|\ l(w) \leq k\}$. Then, for any subset $F \subseteq W$ with $1 \in F$, we define $F_{\leq k} := F \cap W_{\leq k}$. Finally, for any $k \in \mathbb{N}$ and $F \subseteq W$, we define the number $\gamma_{F,k} := |F_{\leq k}|$ and the sequence $\gamma_F := (\gamma_{F,k})_{k\geq 0}$. Then, let $\omega(F) := \omega(\gamma_F) = \limsup_{k\to\infty} \gamma_{F,k}^{1/k}$.

Then, we say that a subset F has exponential growth if $\omega(F) > 1$ and subexponential growth otherwise. If a subset F has subexponential growth, and there is $C \in \mathbb{R}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$ with $\gamma_{F,k} \leq Ck^d$ for all $k \geq 0$, then we say F has *polynomial growth.* If F is of subexponential growth and not of polynomial growth, F has *intermediate growth.*

We note that if $P(F, W) = \sum_{w\in F} u^{l(w)} = \sum_{k\geq 0} a_k u^k$ is the Poincaré series of F, then the coefficients $a_k = \gamma_{F,k} - \gamma_{F,k-1}$ for $k \geq 1$ and $a_0 = \gamma_{F,0}$.

12.2 Properties of $W_{(3)}$

The proof that quotients have exponential growth requires some properties of the universal Coxeter system $W_{(3)} = \langle a, b, c|\ a^2 = b^2 = c^2 = 1\rangle$ with Coxeter diagram



These properties are all known and listed in [29], but we remind the reader here for completeness.

$W_{(3)}$ has the following properties:

1. The Poincaré series of $W_{(3)}$ is $P(W_{(3)}) = \frac{1+q}{1-2q}$.

2. Each $w \in W_{(3)}$ has a unique reduced expression.

3. The conjugacy classes of a, b, and c are all distinct.

4. For $x \neq y \in W_{(3)}/(W_{(3)})_{\{a\}} =: W_{(3)}^{\{a\}}$, we know that $xax^{-1} \neq yay^{-1}$. Also, $P(W_{(3)}^{\{a\}}, W_{(3)}) = \frac{P(W_{(3)})}{1+q} = \frac{1}{1-2q}$ so there are exactly 2^k elements $x \in W_{(3)}^{\{a\}}$ with $l(x) = k$. For each such x, we have $l(xax^{-1}) \leq 2l(x) + 1 = 2k + 1$.

5. Property 4 implies that for $k \geq 0$ there are more than 2^k elements $t \in \{xax^{-1} |\ x \in W_{(3)}^{\{a\}}\}$ with $l(t) \leq 2k + 1$.

6. For any reflection $t \in \{xax^{-1} |\ x \in W_{(3)}^{\{a\}}\}$, properties 2 and 3 imply that a appears in any reduced expression for t.

12.3 Parabolic Quotients

We now prove a generalization of [29, Theorem 1].

Theorem 12.3.1. *Let (W, S) be an irreducible, infinite, non-affine Coxeter system. Then for all $J \subsetneq S$, $W^J = W/W_J$ has exponential growth.*

Our proof is similar to the one found in [29], however, we need to make some changes so that proof works for all Coxeter systems (as opposed to only simply-laced Coxeter systems).

We first recall an important result due to Deodhar, [9].

Theorem 12.3.2. *Let (W, S) be a Coxeter system, T the set of reflections, and $J \subseteq S$. If $t_1, t_2 \in T \setminus W_J$ with $t_1 \neq t_2$, then $t_1 W_J \neq t_2 W_J$, that is, distinct reflections in $T \setminus W_J$ lie in distinct left cosets of W_J.*

According to [29], to prove Theorem 12.3.1, we need to prove the following proposition.

Proposition 12.3.3. *Let (W, S) be an irreducible, infinite, non-affine Coxeter system and let $J \subsetneq S$. Then there exists $M \in \mathbb{N}$ such that for all $k \geq 0$ there are at least 2^k elements $t \in T \setminus W_J$ such that $l(t) \leq M(2k+1)$.*

If this proposition holds, Theorem 12.3.1 holds by the same argument given in [29].

12.4 Proof of Proposition 12.3.3

Proof. For all $x \in W$, any reduced expression for x uses the same simple reflections. We write $\text{supp}(x) = \{r \in S | \ r \text{ appears in all reduced expression for } x\}$. It is also well known that $x \in W_J$ if and only if $\text{supp}(x) \subseteq J$.

Now, let (W, S) be an irreducible, infinite, non-affine Coxeter system and let $s' \notin J$. According to Theorem 11.8.2, (W, S) contains a reflection subgroup W' such that $W' \cong W_{(3)}$. Suppose $\chi(W') = \{t_1, t_2, t_3\}$. Without loss of generality, we may assume that $s' \in \text{supp}(t_1)$. Indeed, if $s' \notin \text{supp}(t_i)$ for all i, then let $Z \subset S$ be the smallest subset such that $W' \subset W_Z$. Then, take a minimal path $s' = s_n - s_{n-1} - \cdots - s_1 - z$ in the Coxeter diagram with $s_i \notin Z$ for all i and $z \in Z$; let the word corresponding to the path be $x = s_1 \cdots s_n \in W$, and by reordering we assume $z \in \text{supp}(t_1)$. Now, we define a sequence of reflections in the following way. Let $r_0 = t_1$ and $r_i = s_i \cdots s_1 t_1 s_1 \cdots s_i$ for all $i \geq 1$. Let $\alpha_{r_i} \in \Phi^+$

be the root corresponding to r_i. Now, since $(\{s_1, ..., s_{i-1}\} \cup Z) \cap (\{s_i\}) = \emptyset$ for all $i \geq 1$ and s_i is connected to s_{i-1}, we have $(\alpha_{r_{i-1}}, \alpha_{s_i}) < 0$ for all i and so $l(r_i) = l(r_{i-1}) + 2$ for all $i \geq 1$. This implies that $l(x^{-1}t_1x) = 2l(x) + l(t_1)$; hence, $s' = s_n \in \text{supp}(x^{-1}tx)$. Then, $xW'x^{-1}$ is a Coxeter system, isomorphic to $W_{(3)}$ with $\chi(xW'x^{-1}) = \{xtx^{-1}, xt_2x^{-1}, xt_3x^{-1}\} = \{t'_1, t'_2, t'_3\}$ with $s' \in \text{supp}(t'_1)$ as required.

Next, by the properties listed in Section 12.2, we know that there are at least 2^k elements $t \in W' \cap T$ such that $l_{(W', \chi(W'))}(t) \leq 2k + 1$ such that $t_1 \in \text{supp}(t)$. Define $M := \max\{l(t_i), l(t_2), l(t_3)\}$. Then, we have $l(w) \leq M(l_{(W', \chi(W'))}(w))$ for all $w \in W'$. This implies that for each $t \in W' \cap T$ above, we have $l(t) \leq M(2k+1)$.

For each $t \in W' \cap T$ described above, t_1 appears in a reduced expression for t; in terms of roots this means $\alpha_t = \sum_{i=1}^{3} c_i \alpha_{t_i}$ with $c_1 > 0$. Since $s' \in \text{supp}(t_1)$, we have $\alpha_{t_1} = \sum_{s \in S} c_s \alpha_s$ with $c_{s'} > 0$. Therefore, it follows that $\alpha_t = \sum_{s \in S} d_s \alpha_s$ with $d_{s'} > 0$ so that $s' \in \text{supp}(t)$. Thus, we have $t \notin W_J$. This implies that $t \in T \setminus W_J$. Therefore, by Theorem 12.3.2, we get that there are at least 2^k elements $t \in T \setminus W_J$ such that $l(t) \leq M(2k + 1)$. The proposition follows. □

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