

An Introduction to Posets and Möbius Inversion

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Introduction

Three Seemingly Unrelated Examples

A Brief Introduction to Posets and Incidence Algebras

The Möbius Function for a Poset

Revisiting the Three Examples

Theory of Finite Differences

Let $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$. In other words, f defines a sequence

$$\{f(m)\}_{m=0}^{\infty} = \{f(0), f(1), f(2), \dots\}.$$

There is an analogue of the integral operator on this space

$$\mathcal{I}(f(n)) = \sum_{i=0}^n f(i)$$

Also, there is an analogue of the derivative operator:

$$\Delta(f(n)) = f(n) - f(n-1); \quad (f(-1) := 0)$$

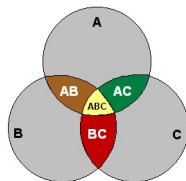
Fundamental Theorem of Difference Calculus:

$$\Delta \mathcal{I}(f(n)) = f(n).$$

Principle of Inclusion/Exclusion

Suppose U is a set with A, B, C all subsets of U .

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$$|U - (A \cup B \cup C)| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

General Case: Let M be a set with subsets M_1, M_2, \dots, M_n . Then

$$\left| M - \bigcup_{i=1}^n M_i \right| = |M| - \sum_{i=1}^n |M_i| + \sum_{1 \leq i < j \leq n} |M_i \cap M_j| + \dots + (-1)^n \left| \bigcap_{i=1}^n M_i \right|$$

Number Theory

Let $\mathbb{Z}_{>0}$ be the set of all positive integers. We define a function $\mu : \mathbb{Z}_{>0} \rightarrow \{-1, 0, 1\}$ in the following way:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime factors} \\ 0 & \text{if } n \text{ has one or more repeated prime factors} \end{cases}$$

Theorem: Suppose that f and g are two functions from $\mathbb{Z}_{>0} \rightarrow \mathbb{R}$ satisfying $f(n) = \sum_{d|n} g(d)$ for all n . Then

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

Unrelated Examples?

- Finite Series (Difference Calculus):

$$\Delta \mathcal{I}(f(n)) = f(n)$$

- Combinatorics (Probability):

$$\left| M - \bigcup_{i=1}^n M_i \right| = \sum_{T \subset \{1, \dots, n\}} (-1)^{|T|} \left| \bigcap_{i \in T} M_i \right|$$

- Number Theory:

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

Posets

Definition

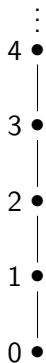
Given a set P , we say a binary relation \leq is a **partial order** on P if it satisfies three properties:

- Reflexivity: $a \leq a$ for all $a \in P$.
- Antisymmetry: If $a \leq b$ and $b \leq a$ then $a = b$.
- Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

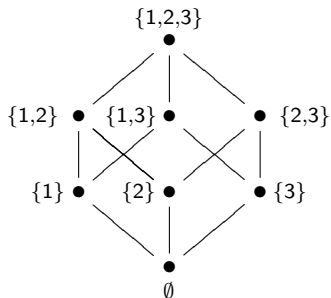
A set with a partial order is called a **poset**.

Examples of Posets

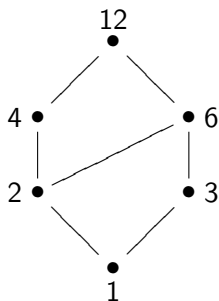
Example ($\mathbb{Z}_{\geq 0}$)



Example (Subsets)



Example (Divisors)



Incidence Algebras on Posets

(P, \leq) is a poset. We define an **interval**:

$$[x, z] = \{\mathbf{y} \in P \mid x \leq \mathbf{y} \leq z\}$$

Incidence algebra:

$$\begin{aligned} I(P) &= \{f : P \times P \rightarrow \mathbb{R} \mid f(x, y) = 0 \text{ unless } x \leq y\} \\ &= \{f : \text{intervals of } P \rightarrow \mathbb{R}\} \end{aligned}$$

- Characteristic function in $I(P)$: $e(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

- Defining function of P in $I(P)$: $\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

Incidence Algebras on Posets

$I(P)$ has two nice operations:

- Addition: pointwise

$$(f + g)(x, z) = f(x, z) + g(x, z)$$

- Multiplication: convolution

$$(f * g)(x, z) = \sum_{x \leq y \leq z} f(x, y)g(y, z)$$

Identity for Multiplication: $e(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

Proposition

A function $f \in I(P)$ is invertible if and only if $f(x, x) \neq 0$ for all $x \in P$.

The Möbius Function of a Poset

Recall: $\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

Question: What is ζ^{-1} ?

Answer: The Möbius function, μ .

Recursive Formula:

$$\mu(x, x) = 1$$

$$\mu(x, z) = - \sum_{x \leq y < z} \mu(x, y)$$

Möbius Inversion Theorem

Theorem

Let (P, \leq) be a (finite) poset and let f, g be elements of $I(P)$.

The following are equivalent:

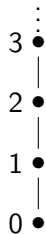
$$(a) \quad f(x, z) = \sum_{x \leq y \leq z} g(x, y) \quad (f = g * \zeta)$$

$$(b) \quad g(x, z) = \sum_{x \leq y \leq z} f(x, y) \mu(y, z) \quad (g = f * \mu)$$

where μ is the Möbius function for the poset.

Theory of Finite Differences

The Chain: $\mathbb{Z}_{\geq 0}$



$$\mu(i, j) = \begin{cases} 1 & j = i \\ -1 & j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

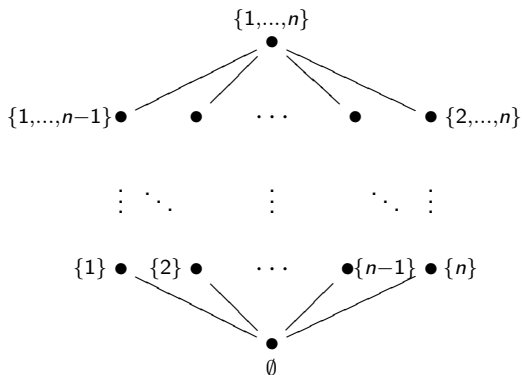
Let $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$. Recall $\mathcal{I}(f(n)) = \sum_{i=0}^n f(i)$.

Möbius Inversion:

$$f(n) = \sum_{i=0}^n \mathcal{I}(f(i)) \mu(i, n) = \mathcal{I}(f(n)) - \mathcal{I}(f(n-1)) = \Delta \mathcal{I}(f(n))$$

Principle of Inclusion/Exclusion

Subsets of $S = \{1, 2, \dots, n\}$



$$\mu(S, T) = \begin{cases} (-1)^{|T|-|S|} & S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

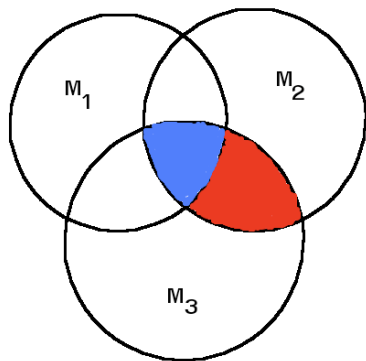
Principle of Inclusion/Exclusion

M a set with subsets M_1, M_2, \dots, M_n .

Let (P, \subseteq) be the poset of subsets of $S = \{1, \dots, n\}$.

We define two functions on the poset (P, \subseteq) :

- $f(V) = |\bigcap_{i \in V} M_i|$
- $g(V) = |\{a \in M \mid a \in M_i \forall i \in V; a \notin M_j \forall j \notin V\}|$



$$g(\{2, 3\}) = \text{Red}$$

$$g(\{1, 2, 3\}) = \text{Blue}$$

$$\text{Note: } f(\{2, 3\}) = g(\{2, 3\}) + g(\{1, 2, 3\})$$

Principle of Inclusion/Exclusion

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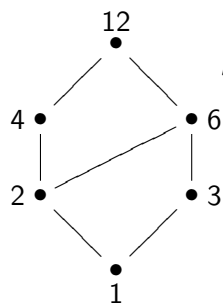
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- $g(V) = |\{a \in M \mid a \in M_i \forall i \in V; a \notin M_j \forall j \notin V\}|$
- $f(V) = \sum_{V \subseteq T} g(T)$
- Remark: $g(\emptyset) = |M - \bigcup_{i=1}^n M_i|$

Möbius Inversion:

$$\begin{aligned} \left| M - \bigcup_{i=1}^n M_i \right| &= g(\emptyset) = \sum_{\emptyset \subseteq T} \mu(\emptyset, T) f(T) \\ &= \sum_T (-1)^{|T|} \left| \bigcap_{i \in T} M_i \right| \end{aligned}$$

Number Theory



$$\mu(1, n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ has repeated prime factors} \end{cases}$$

Remark: $\mu(d, n) = \mu\left(1, \frac{n}{d}\right)$

$$f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R} \text{ with } f(n) = \sum_{d \leq n} g(d)$$

Möbius Inversion:

$$g(n) = \sum_{d \leq n} \mu(d, n) f(d) = \sum_{d \leq n} \mu\left(1, \frac{n}{d}\right) f(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

Thanks!

Any Questions?

References

Bender, E.A and J.R. Goldman, *On the Applications of Mobius Inversion in Combinatorial Analysis*. The American Mathematical Monthly, Vol. 82, No. 8. (Oct., 1975), pp. 789-803.

Cameron, Peter J., *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, 1994.