Differential Equations & Taylor Series

Goal: To find a function $y(x)$ that satisfies $\frac{dy}{dx} = 1 + x \cdot y$, subject to $y(0) = 1$.

Problem: The differential equation $\frac{dy}{dx} = 1 + x \cdot y$ is not ________________, so our "standard method" won't work.

Solution: Use Taylor's Series! Let $y = \sum_{n=0}^{\infty} c_n \cdot x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$ for some appropriate coefficients c_n. Then:

$$\begin{aligned}
\frac{dy}{dx} &= 1 + x \cdot y \\
\frac{dy}{dx} &= 1 + x \cdot \sum_{n=0}^{\infty} \underline{\qquad\qquad\qquad} \\
\sum_{n=0}^{\infty} \underline{\qquad\qquad\qquad} &= 1 + x \cdot \sum_{n=0}^{\infty} \underline{\qquad\qquad\qquad} \\
\sum_{n=0}^{\infty} \underline{\qquad\qquad\qquad} &= 1 + \sum_{n=0}^{\infty} \underline{\qquad\qquad\qquad} \\
\underline{\quad} + \underline{\quad}x + \underline{\quad}x^2 + \underline{\quad}x^3 + \underline{\quad}x^4 + \ldots &= \underline{\quad} + \underline{\quad}x + \underline{\quad}x^2 + \underline{\quad}x^3 + \underline{\quad}x^4 + \ldots
\end{aligned}$$

Since $y(0) = 1$ we know that $c_0 =$ ____. From the equation above, $c_1 =$ ____. After that, each c_n is related to the one two before it, c_{n-2}, by the equation $c_n = c_{n-2}/(\quad)$. This gives us:

$$c_0 = \underline{\quad}, \quad c_2 = \underline{\quad}, \quad c_4 = \underline{\quad}, \quad c_6 = \underline{\quad}, \quad \ldots, \quad c_{2n} = \underline{\qquad\qquad\qquad} \quad \ldots$$

$$c_1 = \underline{\quad}, \quad c_3 = \underline{\quad}, \quad c_5 = \underline{\quad}, \quad c_7 = \underline{\quad}, \quad \ldots, \quad c_{2n+1} = \underline{\qquad\qquad\qquad} \quad \ldots$$

Note that

$$\begin{aligned}
2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n) &= (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \ldots \cdot (2 \cdot n) \\
&= (2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2) \cdot (1 \cdot 2 \cdot 3 \cdot \ldots \cdot n) \\
&= (\qquad) \cdot (\qquad)!
\end{aligned}$$

and that

$$\begin{aligned}
1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) &= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot (2n) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \\
&= \frac{(\qquad)!}{(\qquad) \cdot (\qquad)!}
\end{aligned}$$

so we can write $c_{2n} = \frac{1}{(\quad)\cdot(\quad)!}$ and $c_{2n+1} = \frac{(\quad)\cdot(\quad)!}{(\qquad)!}$. We conclude:

$$y = \left(\sum_{n=0}^{\infty} \frac{(\qquad)}{(\qquad)} \cdot x^{2n} \right) + \left(\sum_{n=0}^{\infty} \frac{(\qquad)}{(\qquad)} \cdot x^{2n+1} \right).$$