

Differential Equations & Taylor Series

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Goal: To find a function $y(x)$ that satisfies $\frac{dy}{dx} = 1 + x \cdot y$, subject to $y(0) = 1$.

Problem: The differential equation $\frac{dy}{dx} = 1 + x \cdot y$ is not _____, so our “standard method” won’t work.

Solution: Use Taylor’s Series! Let $y = \sum_{n=0}^{\infty} c_n \cdot x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$ for some appropriate coefficients c_n . Then:

$$\begin{aligned} \frac{dy}{dx} &= 1 + x \cdot y \\ \frac{dy}{dx} &= 1 + x \cdot \sum_{n=0}^{\infty} \underline{\hspace{2cm}} \\ \sum_{n=0}^{\infty} \underline{\hspace{2cm}} &= 1 + x \cdot \sum_{n=0}^{\infty} \underline{\hspace{2cm}} \\ \sum_{n=0}^{\infty} \underline{\hspace{2cm}} &= 1 + \sum_{n=0}^{\infty} \underline{\hspace{2cm}} \\ \underline{\hspace{1cm}} + \underline{\hspace{1cm}}x + \underline{\hspace{1cm}}x^2 + \underline{\hspace{1cm}}x^3 + \underline{\hspace{1cm}}x^4 + \dots &= \underline{\hspace{1cm}} + \underline{\hspace{1cm}}x + \underline{\hspace{1cm}}x^2 + \underline{\hspace{1cm}}x^3 + \underline{\hspace{1cm}}x^4 + \dots \end{aligned}$$

Since $y(0) = 1$ we know that $c_0 = \underline{\hspace{1cm}}$. From the equation above, $c_1 = \underline{\hspace{1cm}}$. After that, each c_n is related to the one two before it, c_{n-2} , by the equation $c_n = c_{n-2}/(\underline{\hspace{1cm}})$. This gives us:

$$\begin{aligned} c_0 &= \underline{\hspace{1cm}}, & c_2 &= \underline{\hspace{1cm}}, & c_4 &= \underline{\hspace{1cm}}, & c_6 &= \underline{\hspace{1cm}}, & \dots, & c_{2n} &= \underline{\hspace{2cm}} \dots \\ c_1 &= \underline{\hspace{1cm}}, & c_3 &= \underline{\hspace{1cm}}, & c_5 &= \underline{\hspace{1cm}}, & c_7 &= \underline{\hspace{1cm}}, & \dots, & c_{2n+1} &= \underline{\hspace{2cm}} \dots \end{aligned}$$

Note that

$$\begin{aligned} 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n) &= (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \dots \cdot (2 \cdot n) \\ &= (2 \cdot 2 \cdot 2 \cdot \dots \cdot 2) \cdot (1 \cdot 2 \cdot 3 \cdot \dots \cdot n) \\ &= (\underline{\hspace{1cm}}) \cdot (\underline{\hspace{1cm}})! \end{aligned}$$

and that

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1) &= \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n-1) \cdot (2n) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} \\ &= \frac{(\underline{\hspace{1cm}})!}{(\underline{\hspace{1cm}}) \cdot (\underline{\hspace{1cm}})!} \end{aligned}$$

so we can write $c_{2n} = \frac{1}{(\underline{\hspace{1cm}}) \cdot (\underline{\hspace{1cm}})!}$ and $c_{2n+1} = \frac{(\underline{\hspace{1cm}}) \cdot (\underline{\hspace{1cm}})!}{(\underline{\hspace{1cm}})!}$. We conclude:

$$y = \left(\sum_{n=0}^{\infty} \frac{(\underline{\hspace{1cm}})}{(\underline{\hspace{1cm}})} \cdot x^{2n} \right) + \left(\sum_{n=0}^{\infty} \frac{(\underline{\hspace{1cm}})}{(\underline{\hspace{1cm}})} \cdot x^{2n+1} \right).$$