

Math 253
The Dot Product

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, and let θ be the angle between \vec{u} and \vec{v} , with $0 \leq \theta \leq \pi$.

Definition 1: The dot product of \vec{u} and \vec{v} , denoted $\vec{u} \cdot \vec{v}$, is the scalar:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Definition 2: The dot product of \vec{u} and \vec{v} , denoted $\vec{u} \cdot \vec{v}$, is the scalar:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Theorem: Definitions 1 & 2 are equivalent.



Proof: In the above left $\triangle ABC$, the Law of Cosines (LOC) tells us that:

$$c^2 = \text{_____} + \text{_____} - \text{_____} \cos C.$$

We redraw the same triangle above right, but label the sides \vec{u} , \vec{v} , and _____, and the angle of interest θ . Then we get:

$$\|\text{_____}\|^2 = \|\text{_____}\|^2 + \|\text{_____}\|^2 - \text{_____} \cos \theta \quad (\text{LOC})$$

$$(\text{_____})^2 + (\text{_____})^2 + (\text{_____})^2 = \|\text{_____}\|^2 + \|\text{_____}\|^2 - \text{_____} \cos \theta \quad (\text{expand})$$

$$\text{_____} = \|\text{_____}\|^2 + \|\text{_____}\|^2 - \text{_____} \cos \theta \quad (\text{expand})$$

$$\text{_____} = \text{_____} + \text{_____} - \text{_____} \cos \theta \quad (\text{expand})$$

$$\text{_____} = \text{_____} \cos \theta \quad (\text{cancel})$$

$$\text{_____} = \text{_____} \cos \theta \quad (\text{divide by -2})$$

This finishes the proof.

Two vectors \vec{u} and \vec{v} are called orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Theorem: If \vec{u} and \vec{v} are orthogonal, then either: _____

or: _____

Example 1: Which of $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{v} = 2\vec{i} - 2\vec{j} + 2\vec{k}$, and $\vec{w} = \vec{j} + 3\vec{k}$ are orthogonal?

Example 2: Vector \vec{a} has magnitude 6 and \vec{b} has magnitude 2. What can be said about $\vec{a} \cdot \vec{b}$?

Example 3: Let $\vec{v} = \sqrt{3}\vec{i} + \vec{j}$ and $\vec{w} = \vec{i}$. Find $\vec{v} \cdot \vec{w}$ using *both* definitions. Hint: for definition 1, it might help to graph them first.

Example 4: Let $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = 7\vec{i} + \vec{j}$. Find the angle between them.

Example 5: Find a general formula for the angle between \vec{v} and \vec{w} .