Calculus I, Math 151 Class notes #14

3.6. Derivatives of logarithmic functions

Warm up question: Can you use the Power Rule to differentiate $y = x^x$? If not, how could you differentiate it?

- We found in the last section that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$. Similarly, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$. **Derive this formula!**
- Applying the Chain Rule, we get that $\dfrac{d}{dx}[\ln g(x)] = \dfrac{g'(x)}{g(x)}$. Why do we care about this formula? Because it is what allows us to differentiate functions like x^x.
- The list at the bottom of page 218 of the book states the four different cases that can occur with functions of the form f^g:

  1. $\frac{d}{dx}(a^b) = 0$. That is, if both the base and exponent of an expression are constant, the entire expression is a constant, and therefore the derivative is 0.
  2. $\frac{d}{dx}[f(x)^b] = b[f(x)]^{b-1}f'(x)$. That is, if the base is a function and the exponent is a constant, then you use the Power Rule (in combination with the Chain Rule, if necessary).
  3. $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$. That is, we use the rule for differentiation of exponential functions (in combination with the Chain Rule, if necessary).
  4. If both the base and exponent are functions, then we can't use any of the above, and have to use logarithmic differentiation instead.

- You will work on a worksheet to figure out how logarithmic differentiation works on your own. Logarithmic differentiation consists of the following:

  1. If $y = f(x)$, then take the natural logarithm of both sides (natural log because its derivative is the easiest);
  2. Using the properties of logarithms, simplify $\ln f(x)$ (see examples 7 and 8 in book);
  3. Take derivatives of both sides;

4. The expression $\ln y$ has derivative $\frac{y'}{y}$, so we get $y'/y = [\ln f(x)]'$.
5. Finally, $y' = y[\ln f(x)]'$.

- Logarithmic differentiation is used when the function is of the form $f(x)^{g(x)}$ or when it is a product and/or of many functions, and the use of product and quotient rules would be brutally long. The advantage of logarithms is that they change powers into products, and products into sums, which simplifies expressions greatly.

Example: How to find $\frac{d}{dx}x^x$?

$$y = x^x \Leftrightarrow \ln y = \ln x^x = x \ln x \Leftrightarrow (\ln y)' = (x \ln x)' \Leftrightarrow \frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1 \Leftrightarrow y' = y(\ln x + 1) = x^x(\ln x + 1).$$