Initial proof solutions

1. Suppose that a is a real number. Prove that if $a^3 > a$ then $a^5 > a$.

   What we know: $a^3 > a$. What we need: $a^5 > a$. How do we know what to do? One thought: if we were able to show that $a^5 > a^3$, then by transitivity, it will follow that $a^5 > a^3 > a$, which is what we need. Can we show that $a^5 > a^3$? Yes, because if we multiply both sides of the given iinequality by a^2, we get the required inequality.

   Let's put it all together. This is what a formal proof would look like:

   Suppose $a^3 > a$. Notice that a cannot be 0 (because then $a^3 = a$), which means that $a^2 > 0$ (since $a^2 \geq 0$ for all real numbers, and is only 0 when $a = 0$), and we can multiply both sides of the given inequality by a^2 without changing it:

   $$a^3 > a \leftrightarrow a^3 \cdot a^2 > a \cdot a^2 \leftrightarrow a^5 > a^3.$$

   Since "greater than" is a transitive relation, it follows that

   $a^5 > a^3$ (proved above) and $a^3 > a$ (given) implies that $a^5 > a$, which was to be shown.

   Notes: We did not have to prove that you can multiply an inequality by a positive number without changing the sign, or that inequality is transitive, because these are supposed to be known facts. You do have to state them, though. You probably didn't have to explain why $a^2 > 0$ in too much detail. How detailed your proof is depends on what your audience is. It's always better to explain too much than too little. Finally, proofs often end in QED ("Quod Eram Demonstrandum") instead of "which was to be shown."

2. What's wrong with the following proof of the theorem?

   (a) Suppose $x = 7$. Then $\dfrac{2x-5}{x-4} = \dfrac{2(7)-5}{7-4} = 9/3 = 3$. Therefore if $\dfrac{2x-5}{x-4} = 3$, then $x = 7$.
   You are using the conclusion to prove the assumption. The mistake is in thinking that a statement and its converse are equivalent.

   (b) Give a correct proof of the theorem.

   Solve for x. This is a sufficient proof because algebraic solutions are valid in proofs, and you ended up with 7 as the answer.

3. Prove that if $x^2 + y = 13$ and $y \neq 4$, then $x \neq 3$.

   Note that this is a statement of the form $p \wedge \neg q \to \neg r$. We showed in class (see also next problem) that this is equivalent to $p \wedge r \to q$, which is much easier to prove. Just plug in $x = 3$ into $x^2 + y = 13$, and you will get

$y = 4$. Note that this type of formulation often occurs. You may assume from now on that we know that you can exchange $\neg q \to \neg r$ with $r \to q$, even when there is also a p involved like in this problem.

4. Suppose that x and y are real numbers. Prove that if $x^2y = 2x + y$, then if $y \neq 0$, then $x \neq 0$.

   This problem has a confusing logical structure. However, notice that it has the form $p \to (\neg q \to \neg r)$. Also notice the following:

   $$p \to (\neg q \to \neg r) \equiv p \to (r \to q) \equiv \neg p \vee (r \to q) \equiv \neg p \vee (\neg r \vee q) \equiv (\neg p \vee \neg r) \vee q \equiv$$
   $$\neg(p \wedge r) \vee q \equiv (p \wedge r) \to q,$$

   which means that you really need to prove the following statement:

   Prove that if $x^2y = 2x + y$ and $x = 0$, then $y = 0$,

   which is much easier, of course. All you need to do is plug in $x = 0$ and get

   $$0y = 0 + y \leftrightarrow 0 = y$$

   QED.

5. Suppose A, B, and C are sets, and $A \setminus B \subseteq C$. Then $A \setminus C \subseteq B$.

   We did this problem in class. Notice that we only used the logical structure of the statements, rather than their meaning. An alternative proof, one that would use the meaning of the statements, would look like this:

   Let $A \setminus B \subseteq C$, and let $x \in A \setminus C$. We need to show that $x \in B$, by definition of subsets.

   (Whenever you deal with sets, you will use proofs like this one. In particular, if you want to show that $A \subseteq B$, according to the definition of subset: $A \subseteq B$ iff $x \in A \to x \in B$, you will assume $x \in A$ and somehow conclude that $x \in B$. This will prove the claim.)

   Since $x \in A \setminus C$, this means that $x \in A \wedge x \notin C$. Since $A \setminus B \subseteq C$, and since our x is not in C, this means that $x \notin A \setminus B$. Since an element of A is either in $A \setminus B$ or in $A \cap B$, since it's not in the first set, it has to be in the second ("eliminating a possibility"), and so $x \in A \cap B$, and ("in particular") $x \in B$, QED.

6. Prove that for every real number x, if $x > 0$ then there is a real number y such that $y(y + 1) = x$.

   In this case, you can solve for y. It will depend on x, but this is okay, since we are saying that for every x a y exists. This means that y is related to x. If the statement said "exists y such that for all x," then y would need to be independent of x.

   You know that you can solve for y because this looks like the quadratic formula. Note that we can rewrite this as

   $$y^2 + y - x = 0.$$

Solve for y. Remember that if $ax^2+bx+c=0$, then $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, and so since we are solving for y (x is given since the statement says "for all x exists y")

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-x)}}{2} = \frac{-1 \pm \sqrt{1+4x}}{2},$$

and since $x > 0$, there is always a solution (actually two), which proves the claim. Think of it this way: No matter what x I give you, you can put it in the formula and give me a y. Based on how we solved the problem, this y is such that $y(y+1) = x$.

If you wanted to write this up more formally (I am skipping some of the details provided above), you could say something like this:

Let x be an arbitrary positive real number, and let $y = \frac{-1+\sqrt{1+4x}}{2}$ be a solution to the quadratic equation $y^2 + y - x = 0$. This y satisfies the equation $y(y+1) = x$ by construction, which proves the claim.

7. Suppose x is an integer. Prove that x is even iff x^2 is even.

   There are two directions to prove here, since it is an "iff" statement. We will use the following definition of an even integer: an integer is even if it can be written as $2k$, where k is an integer.

   First assume x is even. That means that $x = 2k$, where k is an integer (by definition of even number). Squaring x gives us that $x^2 = (2k)(2k) = 4k^2 = 2(2k^2)$, which, again by definition of even, means that x^2 is even, since it's of the form $2l$, where $l = 2k^2$ is an integer.

   For the other direction, it may be easiest to prove the contrapositive instead. Assume that x is not even. Then $x = 2k+1$, and $x^2 = (2k+1)^2 = 4k^2 + 4k + 1$, which is an odd number. This means that x is odd implies that x^2 is odd, which is equivalent to proving that x^2 is even implies that x is even.

8. Show that $\sqrt{2}$ is not a rational number.

   This is a famous proof. You can look it up online.