Proving and disproving existential and universal statements

To prove an existential statement $\exists x P(x)$, you have two options:

- Find an a such that $P(a)$;
- Assume no such x exists and derive a contradiction.

In classical mathematics, it is usually the case that you have to do the latter. Constructive mathematics does not allow proofs by contradiction, and requires a witness for an existential statement in order to accept it as true. This is much easier with computers, of course.

To disprove an existential statement $\exists x P(x)$, you again have two options:

- Show instead that $\forall x \neg P(x)$, by taking a general x and showing $\neg x P(x)$;
- Assume $P(x)$ holds for some x and get a contradiction.

Universal statements are just the opposite of existential ones (for example, $\forall x P(x)$ is the same as $\neg \exists x \neg P(x)$), so the rules are similar.

To prove a universal statement $\forall x Q(x)$, you can either

- Take a general x and show that $Q(x)$ always holds (you don't use any special properties of x). In particular, if the statement is of the form $\forall x(V(x) \to W(x))$, then you should assume $V(x)$ holds for a general x and try to show that this implies that $W(x)$ holds as well. If you are showing that $\forall x(V(x) \leftrightarrow W(x))$, then you do the same thing, except you also have to show that if $W(x)$ holds for a general x, then so does $V(x)$;
- Assume there is some x for which $Q(x)$ doesn't hold, and get a contradiction.
- Try out all possibilities – this only works if there is a finite number of possibilities.

To disprove a universal statement $\forall x Q(x)$, you can either

- Find an x for which the statement fails;
- Assume $Q(x)$ holds for all x and get a contradiction.

The former method is much more commonly used.

Here are some examples of existential and universal statements. In each case you will have to figure out if the statement is true or false, and then to prove or disprove it. First write the statement using quantifiers.

1. For any two real numbers a and b, if $a^2 = b^2$, then $a = b$.
2. The sum of any two even numbers is an even number.
3. There are two numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

4. For any two real numbers a and b, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.
5. Every positive integer less than 26 can be written as a sum of three or fewer perfect squares.
6. For all real numbers x, if $0 < x < 1$, then $x^2 < x$.
7. For any three sets A, B, and C, $A \cap (B \setminus C) = (A \cap B) \setminus C$.

Mistakes to avoid:

- Making an argument based on a few examples;
- Using the same letter to mean different things;
- Assuming what needs to be proved.