Math 245
March 19, 2010
Induction practice problems

1. For each positive integer n, let $P(n)$ be the formula

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (a) Write $P(1)$. Is $P(1)$ true?
   (b) Write $P(k)$.
   (c) Write $P(k+1)$.
   (d) In a proof by mathematical induction that the formula holds for all integeres $n \geq 1$, what must be shown to be the inductive step?

2. Find the mistake in the following proof fragment.

   Theorem. For any integer $n \geq 1$,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Proof. Certainly the theorem is true for $n = 1$ because $1^1 = 1$ and $\frac{1(1+1)(2\cdot 1+1)}{6} = 1$. So the base step is true.

   For the inductive step suppose that for some integer $k \geq 1$,

$$k^2 = \frac{k(k+1)(2k+1)}{6}.$$

   We must show that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \ldots$$

3. Fill in the missing pieces in the following proof that

$$1 + 3 + 5 \ldots + (2n-1) = n^2$$

   for all integers $n \geq 1$.

   Proof. Let the property $P(n)$ be the equation

$$1 + 3 + 5 \ldots + (2n-1) = n^2.$$

   Show that the property is true for $n = 1$. To establish the property for $n = 1$, we must show that when 1 is substituted in place of n, the left-hand side equals the right-hand side. But when $n = 1$, the left-hand side is the sum of all odd integers from 1 to $2 \cdot 1 - 1$, which is the sum of the odd integers from 1 to 1, which is just 1. The right hand side is ___________, which also equals 1. so the property i s true for $n = 1$.

Show that for all integers $k \geq 1$, if the property is true for $n = k$, then it is true for $n = k + 1$. Let k be any integeer with $k \geq 1$.

Suppose $1 + 3 + 5 + \ldots + (2k - 1) =$ ___________.

We must show that ___________ $=$ ___________.

But the left-hand side of the previous equation is

$$1 + 3 + 5 + (2(k + 1) - 1) = 1 + 3 + 5 + \ldots (2k + 1) \text{ by algebra}$$

$$= [1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) \text{ by } \underline{\hspace{3cm}}$$

$$= (k + 1)^2 \text{ by algebra}$$

which is the right-hand side of the previous equation, as was to be shown.

4. What is wrong with the following "proof" that all horses are the same color?

   Let $P(n)$ be the proposition that all horses in a set of n horses are the same color. Clearly, $P(1)$ is true. Now assume that $P(n)$ is true, so that all horses in any set of n horses are the same color. Consider any $n + 1)$ horses; number these horses 1, 2, 3,dots $n, n + 1$. Now the first n of these horses all must have the same color, and the last n of these horses must also have the same color. Since the set of the first n horses and the set of the last n horses overlap, all $n + 1$ must be the same color. This shows that $P(n + 1)$ is true, which finishes the proof by induction.

5. Find the flaw with the following proof that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

   Base case: $a^0 = 1$ is true by definition of a^0.

   Inductive step: Assume that $a^k = 1$ for all nonnegative integers k with $k \leq n$. Then note that

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

6. Even-numbered problems 2-12 from 4.3.