Name: _____

Math 245, Discrete Structures
April 23, 2010
Test 2

1. State the principle of mathematical induction.

   Why does induction work?

2. What is wrong with the following two proofs?

   (a) **Theorem:** For any integer $n \geq 1$, all the numbers in a set of n numbers are equal to each other.

   Proof: It is obviously true that all numbers in a set consisting of just one number are equal to each other, so the base case is true. For the induction step, let $A = \{a_1, a_2, \ldots a_k, a_{k+1}\}$ be any set of $k+1$ numbers. Form two subsets each of size k:

   $$B = \{a_1, a_2, a_3, \ldots a_k\} \text{ and } C = \{a_1, a_3, a_4, \ldots a_{k+1}\}.$$

   (B contains all the elements of A except a_{k+1}, and C contains all the elements of A except a_2.)

   By the induction hypothesis, all the numbers in B equal a_1 and all the numbers in C equal a_1 (since both sets have only k numbers. But every number in A is in B or C, so all the numbers in A equal a_1; hence, all are equal to each other.

   (b) **Theorem:** For all integers $n \geq 1$, $3^n - 2$ is even.

   Proof: Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But,

   $$3^{k+1} - 2 = 3^k \cdot 3 - 2 = 3^k(1+2) - 2 = (3^k - 2) + 3^k \cdot 2.$$

   Now $3^k - 2$ is even by induction hypothesis, and $3^k \cdot 2$ is even because it has 2 as a factor. Hence the sum of the two quantities is even. It follows that $3^{k+1} - 2$ is even, and therefore $3^n - 2$ is even for all integers n.

3. Let (a_n), where $n \in \mathbb{N}$ be defined as follows:

$$a_0 = 1,\ a_k = \frac{a_{k-1}}{1 + a_{k-1}},$$

for all integers $k \geq 1$. Find the first few terms of the sequence and guess the general formula for a_n. Then prove the formula using induction.

4. Show that the following is an equivalence relation:

For all $x, y \in \mathbb{R}$, xRy iff $x - y$ is an integer.

What do the equivalence classes of this relation look like? Give some examples.

5. Suppose that in a certain state all automobile license plates have four letters followed by three digits.

   (a) How many different license plates are possible?

   (b) How many license plates could begin with TGIF?

   (c) What is the probability that a licence plate begins with TGIF?

   (d) How many license plates are possible in which all the letters and digits are distinct?

   (e) In 2006, there were 250,844,644 registered vehicles in the US. If a uniform length of a licence place was required, and both letters and numbers were allowed in all positions, what should the number of symbols in a licence plate be to ensure that all vehicles get a unique licence plate, but that the number of options isn't too large? (Hint: guess and check would be a perfectly valid strategy in this problem.)

6. The below text has been taken from http://www.nanpa.com/area_codes/

"The format of an area code is NXY, where N is any digit 2 through 9 and X and Y are any digit 0 through 9. Initially, the middle digit of an area code had to be "0" or "1". When this restriction was removed in 1995, additional area code combinations became available. There are 800 possible combinations associated with the NXY format. Some of these combinations, however, are not available or have been reserved for special purposes

Among them are the following:

Easily Recognizable Codes When the second and third digits of an area code are the same, that code is called an easily recognizable code (ERC). ERCs designate special services; e.g., 888 for toll-free service.

N11 These 8 ERCs, called service codes, are not used as area codes.

N9X The 80 codes in this format, called expansion codes, have been reserved for use during the period when the current 10-digit NANP number format undergoes expansion.

37X and 96X Two blocks of 10 codes each have been set aside by the INC for unanticipated purposes where it may be important to have a full range of 10 contiguous codes available."

(a) How many area codes were available before 1995? Assume there were no other restrictions other than ones mentioned in the first paragraph.

(b) Verify the figures in the text (800, 8, 80, and 10) using combinatorics.

(c) How many area codes are in use?

7. What is the difference between a combination and a permutation? Give examples.