Math 245, Spring 2010, Weekly homework, due on February 26

Note that *conjunction* means “and” and *disjunction* means “or.”

1. Show that $p \to q$ is logically equivalent to $\neg p \vee q$.

   Use a truth table.

2. Show that implication is not associative, i.e. that $(p \to q) \to r$ is not logically equivalent to $p \to (q \to r)$. You may use the previous problem, but it is not necessary.

   You can either use a truth table, or the following chain of equivalences:

$$(p \to q) \to r \equiv \neg(p \to q) \vee r \equiv \neg(\neg p \vee q) \vee r \equiv (p \wedge \neg q) \vee r$$

$$p \to (q \to r) \equiv \neg p \vee (q \to r) \equiv \neg p \vee (\neg q \vee r).$$

   It is easy to come up with a valuation for p, q, r which is true for one but not the other formula: since it is easier to make a disjunction true, look at $\neg p \vee \neg q \vee r$. Let p be false, q be false and r be false. then $\neg p \vee \neg q \vee r$ is true, but $(p \wedge \neg q) \vee r$ is false, so the two formulas cannot be equivalent.

3. A formula is in a disjunctive normal form if it is a disjunction of conjunctions. For example, $(p_1 \wedge p_2) \vee (p_1 \wedge p_3)$ is in disjunctive normal form. Every logical formula, regardless of the number of variables, can be written in disjunctive normal form.

   (a) Convert the formula $(p \to q) \to r$ into disjunctive normal form. Use problem 1 and De Morgan’s laws.

   $(p \wedge \neg q) \vee r$ is in CNF. See problem 2.

   (b) Another way to find a disjunctive normal form of a formula is through truth tables. Make the truth table for $(p \to q) \to r$, and underline the rows for which this statement is true. For example, when p is false AND q is false AND r is true, the whole statement is also true, which you can write as $(\neg p \wedge \neg q \wedge r)$. Find all such conjunctions, and connect them with “ors.” This gives you a disjunctive normal form. Can you explain why this formula is equivalent to the original formula?

$$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r).$$

   This is equivalent to the original formula because it will be true exactly when the original formula is true and false otherwise.

   (c) (extra credit) Use the rules from Section 1.1 to show that the two disjunctive normal forms we obtained are equivalent. (You will want to use the absorption rules).

This one is tricky. Of course, it is obvious that the two formulas are equivalent when they have the same truth values. Algebraically, though, we simplify as follows:

$$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r) \equiv$$

$$[(\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge r)] \vee [(\neg p \wedge q \wedge r) \vee (p \wedge q \wedge r)] \vee [(p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r)] \equiv$$

(in the previous line I used two facts – that disjunction is commutative and associative, so I am grouping the terms as I need them, and I am also using the fact that $a \vee a \equiv a$, so I can add copies of a formula to combine terms)

$$(\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge \neg q) \equiv$$

(in the previous row I used the distributive property: $(a \wedge b) \vee (a \wedge \neg b) \equiv a \wedge (b \vee \neg b) \equiv a \wedge T \equiv a$)

$$r \vee (p \wedge \neg q)$$

(distributive property again).

(d) What would a conjunctive normal form look like?

It would be a conjunction of disjunctions.

(e) Find out how Maple can help you find the CNF and DNF. Use it to check your answer from (b).

Maple returns the same DNF, just in different order, as the one from part (c).

4. Show that there are sixteen different binary connectives by writing out a truth table for each one. Label the ones that look familiar (e.g. "and," "iff" etc.)

   Go to http://en.wikipedia.org/wiki/Logical_connective

   There are 16 binary connectives, because a connective is determined by its truth table. Since a truth table for a binary connective has 4 rows, and since there are two options (0 and 1) for each row, there are $2^4 = 16$ different truth tables.

5. A set of connectives is said to be *complete* if all other logical connectives can be expressed from elements of that set. Show that $\{\neg, \wedge\}$ is a complete set of connectives. You can either use the previous problem and logical equivalences (problem 1, De Morgan's Laws, etc), or conjunctive normal forms.

   Notice from the previous problem that all connectives can be expressed in terms of $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$, so it is enough to show that all these can be expressed using only $\neg$ and $\wedge$.

Notice that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ and $p \to q \equiv \neg p \vee q$, so the only connective left to eliminate is $\vee$, and

$$p \vee q \equiv \neg(\neg p \wedge \neg q),$$

by DeMorgan's laws.

6. Connectives commonly used in computer science are *nand* ("not and", $\uparrow$) and *nor* ("not or" $\downarrow$). Show that $\uparrow$ is complete, by showing that $\neg$ and $\wedge$ can be represented using $\uparrow$.

   Notice that by the previous problem we only need to show that $\neg$ and $\wedge$ can be expressed using $\uparrow$.

   If you look at the truth table for $\uparrow$, you notice that $p \uparrow q$ is false when both p and q are true, and true when both p and q are false. This means that $p \uparrow p$ gives us $\neg p$.

   Now, $p \wedge q$ is easy: since $p \uparrow q \equiv \neg(p \wedge q)$, it means that $\neg(p \uparrow q) \equiv p \wedge q$, but we know how to negate a statement – we just "nand" it with itself:

$$p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q).$$