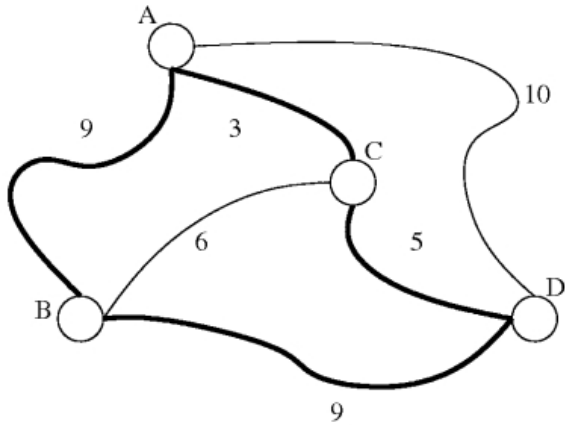


**Math 245**  
**Graph Theory**

- Note that graphs do not be represented using diagrams. Another way is to list all the vertices and all the edges. For example, we can let  $V = \{u_1, u_2, u_3, u_4\}$  be the set of all vertices, and  $E = \{\{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_3, u_4\}\}$  to be the set of all edges. Then we can think of  $E$  as a relation. This makes sense, because earlier in the course we described relations using graphs.
  - Assuming that a graph is simple (no loops, no multiple edges), which of the properties of relations hold for  $E$ ? Is this relation reflexive? symmetric? antisymmetric? transitive?
  - What if we allow loops?
  - What if the graph is directed?
- Suppose we associate a graph  $G$  with a college class in the following manner. The vertices of  $G$  correspond to the students in the class, while two vertices of  $G$  are adjacent if and only if they correspond to two students having the same major. Can you describe the appearance of  $G$ ?
- Give an example of a situation that could be better represented using a digraph than a graph.
- Show that a graph can't exist with vertices of degrees 2, 3, 4, 4, and 5.
  - Show that a graph can't exist with vertices of degrees 1, 3, 3, and 3.
- Let  $m$  and  $n$  be nonnegative integers such that  $m \neq n$ . Find an example of a graph  $G$  such that each vertex has degree  $m$  or  $n$ . Note: Your example must be general. Do not assign specific values to  $m$  and  $n$ .
- (\*) A graph on  $p$  vertices ( $p \geq 2$ ) is called *perfect* if no two of the vertices have equal degrees. Prove that "no graph is perfect."
- A graph is *connected* if there is a path between any two vertices. A *component* of a graph that is not connected is its subgraph that is connected. Give an example of a connected graph  $G$  containing a vertex  $v$  such that  $G - v$  has four components.
- An edge of a graph is a *bridge* if removing it makes the graph disconnected. Let  $G$  be a connected graph containing only even vertices. Prove that  $G$  cannot contain a bridge.
- Give an example of a graph on 5 vertices 5 such that every edge is a bridge.
- The floor plan (attached) is the third home of the billionaire Count Van Diamond. He has just been murdered, and James Bomb, the internationally renowned detective and part time graph theorist, has been called in to investigate. The butler claims he saw the gardener enter from the outside and then, shortly after, leave that room by the same door. The gardener, however, says that he cannot be the man whom the butler saw, for he entered the house, went through each door exactly once, and then left the house. James Bomb checks the floor plan. Within the matter of hours, he declares the case solved. Who killed the Count?
- True or false? Every eulerian graph is hamiltonian.
- True or false? Every hamiltonian graph is eulerian.
- A salesman is traveling locally to sell his product. Below is a diagram of towns he needs to visit, with distances between them.



- (a) Try to find the shortest distance he will need to travel. Assume he will have to return to where he started from. If you wanted to be sure you got the shortest route, how many different circuits would you need to consider? This is called the *brute force* algorithm.
- (b) Now consider the *nearest neighbor* algorithm:
1. stand on an arbitrary vertex as current vertex.
  2. find out the lightest edge connecting current vertex and an unvisited vertex V.
  3. set current vertex to V.
  4. mark V as visited.
  5. if all the vertices in domain are visited, then terminate.
  6. Go to step 2.
- Use this algorithm to find a circuit. Is it the shortest one?
- (c) Now consider the *greedy algorithm*:
- Starting with the least cost edge, look at the edges one by one and select an edge only if the edge, together with already selected edges,
1. does not cause a vertex to have degree three or more
  2. does not form a cycle, unless the number of selected edges equals the number of vertices in the graph.
- Use this algorithm to find a circuit. Is it the shortest one?
- (d) Note that in the Traveling Salesman Problem you either have to sacrifice speed or accuracy. In cases of large graphs, it is not even possible to consider all options. For example, if you had a graph on 100 vertices, how many different circuits would you need to consider?

14. A traveling salesman must visit all the eight cities below and return to where he started from. Use both the greedy and nearest neighbor algorithm to find a route. Do you think it is the least expensive one?

	1	2	3	4	5	6	7	8
1. Chicago, IL		672	848	1737	954	1195	1850	906
2. Philadelphia PA	672		255	2383	1368	1047	2520	1572
3. Boston MA	848	255		2489	1617	1270	2689	1753
4. Seattle WA	1737	2383	2489		1891	2732	708	1033
5. Houston TX	954	1368	1617	1891		967	1615	871
6. Miami FL	1195	1047	1270	2732	967		2565	1710
7. Palo Alto CA	1850	2520	2689	708	1615	2565		952
8. Denver CO	906	1572	1753	1033	871	1710	952	

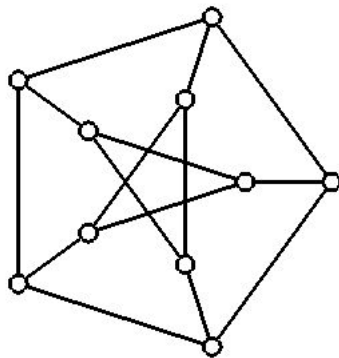
15. A man and his wife are planning a dinner party, but they each have certain peculiarities. The man likes to have everyone at the dinner table know each other, for he feels this creates more harmony during the meal. On the other hand, his wife thinks that no two people at the dinner table should know each other, for she believes one important aspect of dinner parties is the opportunity it gives people for making new acquaintances. Despite this difference of opinion between husband and wife, they are a happily married couple, and both will be happy if either's wishes are fulfilled. Hence, several tables are set, two of which are designated  $A$  and  $B$ . At table  $A$ , the man's table, all guests are to know one another; while at table  $B$ , the wife's table, no two guests know one another. What is the least number of people that may attend a dinner party so that at least one of the two tables can be filled according to these rules?

Ramsey's theorem states that for any pair of positive integers  $(r, s)$ , there exists a least positive integer  $R(r, s)$  such that for any complete graph on  $R(r, s)$  vertices, whose edges are colored red or blue, there exists either a complete subgraph on  $r$  vertices which is entirely blue, or a complete subgraph on  $s$  vertices which is entirely red. Here  $R(r, s)$  signifies an integer that depends on both  $r$  and  $s$ . It is understood to represent the smallest integer for which the theorem holds. Though bounds on Ramsey numbers have been found, the exact values for  $R(m, n)$  have been found for only a few values of  $m$  and  $n$ .

- (a) How is this vertex coloring related to our problem?  
 (b) Show that  $R(3, 3) = 6$ . What does this mean for our problem?  
 (c) Show that  $R(n, 2) = n$ .
16. Under what conditions can a traffic system with all two-way streets be changed to all one-way streets so that in the resulting system, it is possible to travel from any intersection to any other intersection?

a digraph  $D$  is strongly connected if for every two distinct vertices  $u$  and  $v$  of  $D$ , there exists a  $u - v$  path in  $D$  and a  $v - u$  path in  $D$ . A connected graph is called *orientable* if it is possible to assign a direction to each edge of  $G$  to produce a strongly connected digraph  $D$ .

**Theorem:** A connected graph is orientable iff  $G$  contains no bridges.



Show that the above graph (called Petersen graph) is orientable. Assuming the traffic system can be represented with the Petersen graph, show what the system of one-way streets will look like.

17. Can we connect three utilities, water, gas and electricity to each of three houses  $A$ ,  $B$  and  $C$ , so that no pipes cross each other?

A *planar* graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A graph is a *plane* graph if it is already drawn in the plane so that no two edges intersect.

A plane graph  $G$  divides the plane into regions.

**Theorem:** If  $G$  is a connected plane graph with  $p$  vertices,  $q$  edges, and  $r$  regions, then

$$p - q + r = 2.$$

**Note:** this theorem is proved by induction.

- (a) Show that  $K_{3,3}$  is not planar. How is this related to our problem?
  - (b) Show that  $K_5$  is not planar.
  - (c) If there were five houses and two utilities, could we connect the houses with utilities with no intersections?
18. Suppose you are the department chair in your department, and one of your responsibilities is to set up the schedule of classes for next semester. You need to be careful not to have two classes meet at the same time if students are likely to take both classes in the same semester. On the other hand it is most convenient to set up a schedule requiring the fewest number of time periods during the day, since then it might be possible to eliminate classes meeting at unpopular times. The question is: What is the minimum number of hours needed for such a schedule?

This problem is related to graph coloring.

By a *coloring* of a graph  $G$  we mean the assignment of colors to the vertices of  $G$ , one color to each vertex, such that adjacent vertices are assigned different colors. The *chromatic number* of a graph  $G$ , denoted  $\chi(G)$  is the minimum value  $n$  for which an  $n$ -coloring (coloring with  $n$  colors) exists.

**Theorem:** If  $\Delta(G)$  is the maximum degree among the vertices of  $G$ , then  $\chi(G) \leq 1 + \Delta(G)$ .

- (a) How is a scheduling problem related to graph coloring?
- (b) What is the chromatic number of  $K_p$ ?
- (c) What is the chromatic number of a cycle?
- (d) What is the chromatic number of a tree?
- (e) Can a graph have chromatic number 1?
- (f) What is the chromatic number of  $K(n, n)$ ?
- (g) A mathematics department chair plans to offer seven graduate courses next semester, namely combinatorics (C), group theory (G), linear programming (L), numerical analysis (N), probability (P), statistics (S), and topology (T). The mathematics graduate students and the courses they plan to take are:

Android: C, L, T	Ginny: N, P
Bob: C, G, S	Homer: G, L
Carol: G, N	Inga: C, T
Don: C, L	Janet: C, S, T
Ed: L, N	Ken: P, S
Fred: C, G	Linda: P, T

How many time periods are needed for these seven courses?

Note: The recreational puzzle Sudoku can be seen as completing a 9-coloring on given specific graph with 81 vertices.

Note: One of the most famous results in graph theory is the Four Color Theorem, that says that if  $G$  is a planar graph, then it can be colored with at most four colors.

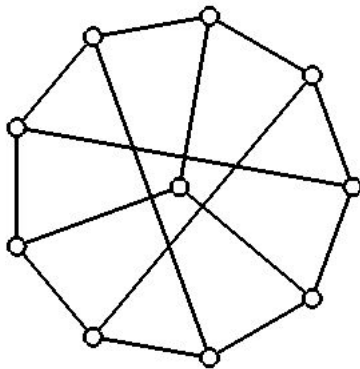
Note: The five color theorem is relatively easy to prove. You will have the proof of this theorem assigned as homework.

19. Much of graph theory can be done using matrices.

Given a graph  $G$  with  $n$  vertices  $v_1, \dots, v_n$ , we define the *adjacency* matrix of  $G$  with respect to the enumeration  $v_1, \dots, v_n$  of the vertices as being the  $n \times n$  matrix  $A = [a_{ij}]$  defined by  $a_{ij} = 1$  if there is an edge between  $v_i$  and  $v_j$  and 0 otherwise.

Note that for an undirected graph the adjacency matrix is symmetric (that is it is equal to its transpose), but it is not necessarily the case for a digraph.

The *incidence* matrix of the graph  $G$  is a matrix  $B = [b_{ij}]$  such that  $b_{ij} = 1$  if  $v_i$  is incident with  $e_j$  and 0 otherwise. In general this matrix is not square.



- Find the adjacency matrix  $A$  for the graph above.
- Find the incidence matrix for the graph above.
- Find  $A^2$ . Show that this matrix gives you the total number of paths of length 2. What do you think  $A^n$  tells you?
- Show that if the incidence matrix of a graph  $G$  is a square matrix, then  $G$  contains a cycle.

Most of these problems have been taken from *Introductory Graph Theory* by Gary Chartrand.