Properties of L_p spaces in subsystems of second-order arithmetic (DRAFT)

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Abstract

Definitions and methods used in proving theorems of measure theory within second-order arithmetic differ from those in standard mathematical practice. Furthermore, it is often possible to give multiple definitions of a term, so the question presents itself of which ones to use. In this paper we address some of these issues. We discuss definitions of L_p spaces and of products and powers within these spaces, as well as their properties. It is argued that the class of *essentially bounded functions* is especially convenient to work with. We also prove some classical results such as Hölder's inequality, a stronger version of the monotone convergence theorem and Fatou's lemma.

1 Preliminaries

The classical theory of measure is highly nonconstructive. Bishop and Bridges state in [2]:

> Any constructive approach to mathematics will find a crucial test in its ability to assimilate the intricate body of mathematical thought called measure theory, or the theory of integration.

Simpson writes in [7], X.1:

> Historically, the subject of measure theory developed hand in hand with the nonconstructive, set-theoretic approach to mathematics.... Although Reverse Mathematics is quite different from Bishop-style constructivism, we see that Bishop's remark raises an interesting question: *Which nonconstructive set existence axioms are needed for measure theory*?

This paper partially answers that question.

The main goal of reverse mathematics is twofold: to prove theorems of countable (non set-theoretic) mathematics within second-order arithmetic, and to determine the exact strength of axioms used in the proofs. The motivation for this approach is primarily foundational, though it should be noted that it also provides insight into the structure and complexity of proofs and the mathematical concepts involved.

I will assume that the reader is familiar with the underlying framework, and will provide only those definitions that are particularly pertinent to the rest of the paper, or those not found in other literature. For a more detailed general account of the subject, consult Simpson's monograph [7]. Simpson and his students (principally Yu), have done a significant amount of work in measure theory. These results can be found in [3], [11], [9], [10]. All results presented below can be found in [6].

The language of second-order arithmetic is two-sorted, consisting of two types of variables: number and set variables. The former are intended to range over natural numbers, and the latter over sets of natural numbers. Full second-order arithmetic consists of basic axioms (propositional logic and axioms for an ordered semiring), the induction axiom and the comprehension scheme

$$\exists X\ \forall n\ (n \in X \leftrightarrow \varphi(n))$$

where φ is any formula of the language in which X doesn't occur freely. In most cases full comprehension is more than is needed to prove a theorem, and can be replaced with weaker set existence axioms. This yields different subsystems of second-order arithmetic. Conveniently, six suffice to prove most theorems of countable mathematics. In this paper, we will only need three:

- [RCA_0]. Basic axioms, plus Σ_1 induction and Δ_1 comprehension.
- [WWKL_0]. RCA_0, plus weak-weak König's Lemma ($WWKL$): if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then $\lim_n \frac{|\{\sigma\in T \mid lh(\sigma)=n\}|}{2^n} = 0$.
- [ACA_0]. RCA_0 plus arithmetic comprehension axiom (ACA).

ACA_0 is a natural system to work with in measure theory, as it provides us with least upper bounds and greatest lower bounds of sequences, assuring existence of measure for countable unions and intersections of integrable sets. However, WWKL_0 turns out to be sufficient for a large portion of discourse about integrable functions, and if one is careful with definitions, it is possible to do a significant amount of work even in RCA_0. In fact, it is usually possible to formalize standard definitions or find equivalent ones in this system. Though it can sometimes be shown that desired properties of these definitions hold in RCA_0, it is oftentimes necessary to reason about pointwise properties of functions, and in those cases stronger axioms, like ($WWKL$) or (ACA) are employed.

In reverse mathematics, we are also interested in *reversals* of theorems: theorems are shown to be logically equivalent to set existence axioms. For our purposes, we will only need the following characterization.

Proposition 1.1 (RCA_0) *The following statements are equivalent:*

1. *Every increasing sequence $\langle a_n \rangle$ of real numbers in $[0,1]$ has a limit.*
2. *If $\langle b_n \rangle$ is any sequence of nonnegative reals such that for each n, $\sum_{i<n} b_i \leq 1$, then $\sum_n b_n$ exists.*
3. *(ACA).*

Unless otherwise specified, all definitions are in RCA_0. Keep in mind that, due to the nature of the language, all objects considered are countable or are completions of countable objects, and are coded with natural numbers. That is, all the objects we deal with are actually codes for those objects.

A (code for a) *complete separable metric space* $\hat{A}$ is presented as a nonempty set $A \subseteq \mathbb{N}$ together with a sequence of real numbers $A \times A \to \mathbb{R}$ such that

$$d(a,a) = 0, \quad d(a,b) = d(b,a), \quad d(a,b) + d(b,c) \geq d(a,c)$$

for $a, b, c \in A$.

A point of $\hat{A}$ is a sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ such that $d(a_n, a_m) < 2^{-m}$ for $m < n$. If a and b are two points in $\hat{A}$, given as $a = \langle a_n \rangle$ and $b = \langle b_n \rangle$, then $d(a,b) = \lim_n d(a_n, b_n)$, and two points a and b are considered equal if $d(a,b) = 0$, hence d is a metric on $\hat{A}$. Each $a \in A$ is identified with the point $\langle a \mid n \in \mathbb{N} \rangle \in \hat{A}$, and A is dense in $\hat{A}$.

A *compact metric space* is a complete separable metric space $\hat{A}$ such that for each $j \in \mathbb{N}$, there is an 2^{-n}-net of points from $\hat{A}$.

A countable vector space A over a countable field K consists of a set $|A| \subseteq \mathbb{N}$ with operations $+ : |A| \times |A| \to |A|$ and $\cdot : |K| \times |A| \to |A|$, and a distinguished element $0 \in |A|$ such that $(|A|, +, \cdot)$ satisfies the usual properties of a vector space over K.

A (code for a) *separable Banach space* $\hat{A}$ consists of a countable vector space A over $\mathbb{Q}$ together with a sequence of real numbers $\| \cdot \| : A \to \mathbb{R}$ such that

1. $\|q \cdot a\| = |q| \, \|a\|$ for all $q \in \mathbb{Q}$ and $a \in A$,
2. $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.

A point of $\hat{A}$ is a sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ of points in A such that $\|a_m - a_n\| \leq 2^{-m}$ for all $m < n$. The function $d(a,b) = \|a - b\|$ is a pseudometric on A. When d is be extended to all of $\hat{A}$, and two points identified if $d(a,b) = 0$, the resulting space is a complete separable normed space.

Let X be a compact metric space. *The Banach space C(X)*, elements of which are functions with a modulus of uniform continuity, is the completion of the countable set $S(X)$ of simple functions over X under the sup norm. More precisely, let $X = \hat{A}$ and d a metric on X. Let $B = A \times \mathbb{Q}^+ \times \mathbb{Q}^+$. Each $b = (a, r, s) \in B$ is intended to represent a function ϕ_b as follows:

$$\phi_b(x) = \begin{cases} 1 & d(a,x) \leq x \\ \frac{r-d(a,x)}{r-s} & s < d(a,x) < r \\ 0 & d(a,x) \geq r. \end{cases}$$

Define $C = \mathbb{Q} \times B$. If $c = \{(q_1, b_1), \ldots (q_n, b_n)\}$ is a finite subset of C, define $\phi_c : X \to \mathbb{R}$ by $\phi_c(x) = \sum_{k=1}^{n} q_k \phi_{b_k}(x)$.

Let $S(X) = \{c \mid c \text{ is a finite subset of } C\}$. Finally, $C(X) = \hat{S}(X)$ under the sup norm given by $\|c\| = \sup_{x \in X} |\phi_c(x)|$.

The elements of $S(X)$ will be referred to as *simple functions*, disregarding the distinction between functions and their codes, as is customary when working in second-order arithmetic. Notice that elements of $C(X)$ are bounded, i.e. if $f \in C(X)$, then there is some constant $M_f \in \mathbb{R}$ such that $|f(x)| \leq M_f$ for all x.

If X is a compact metric space, a *Borel measure* μ is a nonnegative bounded linear functional $\mu : C(X) \to \mathbb{R}$ such that $\mu(1) = 1$.

In the case $X = [0, 1]$, a natural interpretation for μ is the Lebesgue measure, where for $f \in C([0, 1])$, $\mu(f) = \int_0^1 f(x)dx$.

An *open set* O is presented as the union of a countable sequence of balls, with rational radii, centered at points in A. A *closed set* is presented as the complement of an open set. A G_δ set is coded with a sequence of open sets (meaning that it is the intersection of this sequence). It can be assumed that this sequence is decreasing. Let G be a G_δ set coded with a sequence $\langle U_n \rangle$. If $\mu(U_n) \leq 2^{-n}$ for all n, then G is a *null* G_δ set. The complement of a null G_δ set is a *full* F_σ *set*. *Almost everywhere convergence* is defined as convergence outside of a null G_δ set (or equivalently, convergence on a full F_σ set).

If M is a Borel set of finite rank, ACA_0 proves that there exists a G_δ set G such that $M \subseteq G$ and $\mu(G \setminus M) = 0$. This implies that every null set that is "simple enough", that is, at a low level in the Borel hierarchy, is contained in a null G_δ set. This will allow us to prove a.e. convergence of a sequence of functions even if the set on which the sequence converges is not F_σ. The fact follows immediately from Yu's observation in [9] that ACA_0 proves measure for Borel sets of finite rank to be well-defined and regular. For, if $\mu(M) = \inf\{\mu(U) \mid M \subseteq U \text{ and } U \text{ is open}\}$, then there exists a sequence U_n of open sets such that $M \subseteq U_n$ and $\mu(U_n) \leq \mu(M) + 2^{-n}$ for all n. The set $G = \cap_n U_n$ is the required G_δ set.

To discuss integrable functions, we utilize an approach similar to Bishop's, modifying the axioms for the Daniell integral. Functions are basic objects.

Let $p \geq 1$ be a real number and X a compact metric space.

The Banach space $L_p(X)$ is the completion of $S(X)$ under the L_p norm: $\|f\|_p = \mu(|f|^p)^{1/p}$. A function $f \in L_p(X)$ is a strong Cauchy sequence with respect to the norm of functions in $S(X)$. In other words, $f = \langle f_n \mid n \in \mathbb{N} \rangle$ and $\|f_m - f_n\|_p \leq 2^{-m}$ for $n > m$. It is convenient to sometimes assume that the $\langle f_n \rangle$ is a sequence of functions in $C(X)$ and not $S(X)$.

The definition works for all real $p > 1$, though this fact is not obvious when p is not a natural number. It can be shown that if $f \in S(X)$, then $|f|^p \in C(X)$ by showing it has a modulus of uniform continuity, so the definition above makes sense. The case of special interest is when $p = 1$: the elements of $L_1(X)$ are called integrable functions. We sometimes write $\mu(f) = \int f d\mu$ and usually omit the subscript in the norm. It is important to point out the following result, which can be found in [9] and [3].

Proposition 1.2 ($\mathsf{WWKL_0}$) *If f is an element of $L_1(X)$, represented as $\langle f_n \rangle$, then $f(x)$ is defined a.e. and is equal to* $\lim f_n(x)$.

This proposition explains why a large portion of our work goes through in $\mathsf{WWKL_0}$: to discuss pointwise properties of integrable functions, ($WWKL$) often suffices. It will later be shown that $L_p(X) \subseteq L_1(X)$ for all $p > 1$, so elements of all L_p spaces are pointwise defined in $\mathsf{WWKL_0}$.

The classical notion of "measurable" doesn't lend itself to characterization by means of Cauchy sequences, and can, in fact, be bypassed all together.

The following statements will be used in a number of results below. The proof of this proposition can be found in [10]:

Proposition 1.3 ($\mathsf{WWKL_0}$) *For $f, f' \in L_1(X)$ the following properties hold:*

1. *$\|f - f'\| = 0$ if and only if $f = f'$ a.e.*
2. *If $f \leq f'$ a.e, then $\mu(f) \leq \mu(f')$.*

From now on, "$f = g$ in $L_1(X)$" and "$f = g$ a.e. " will be used interchangeably when the underlying system is $\mathsf{WWKL_0}$ or $\mathsf{ACA_0}$.

It is important to keep in mind that functions in $L_p(X)$ are elements of a Banach space. Their pointwise properties are not known in $\mathsf{RCA_0}$ and $f = g$ means, by definition, equality in the normed space: $f = g \leftrightarrow \|f - g\|_p = 0$. Also, as functions are given via representations, it is sometimes necessary to ask the question whether a result or property depends on the sequence that represents the function; independence of representations is an important factor in determining the validity of definitions.

If $f, g \in L_p(X)$, then it is possible to define $\max(f, g)$ and $\min(f, g)$. The former is represented as $\langle \max(f_n, g_n) \rangle$ and the latter as $\langle \min(f_n, g_n) \rangle$ and it is easily shown that they are elements of the space $L_p(X)$. Furthermore, if $f = f'$ and $g = g'$, then $\max(f, g) = \max(f', g')$ and $\min(f, g) = \min(f', g')$. In particular, for $f \in L_p(X)$, we can define $f^+ = \max(f, 0)$, represented as $\langle \max(f_n, 0) \rangle$, and $f^- = \max(-f, 0)$, with the representation $\langle \max(-f_n, 0) \rangle$. It immediately follows (in $\mathsf{RCA_0}$) that $f^+, f^- \in L_p(x)$ and $f = f^+ - f^-$. As $|f| = f^+ + f^-$, $|f|$ is integrable whenever f is.

The following definitions are substitutes in $\mathsf{RCA_0}$ for the usual, pointwise definitions of properties in question.

A function $f \in L_p(X)$ is *nonnegative* if $|f| = f$ in $L_p(X)$.

- Equivalently, f is nonnegative if and only if $f = f^+$ if and only if $f^- = 0$.
- [$\mathsf{RCA_0}$]: It can be assumed that if $\langle f_n \rangle$ is a nonnegative function, then $f_n(x) \geq 0$ for all n and all x, because otherwise f_n can be replaced by f_n^+.
- [$\mathsf{WWKL_0}$]: A function f is nonnegative in the sense defined above if and only if $f(x) \geq 0$ a.e.

Similarly, the $\geq$ relation between two functions in $L_p(X)$ can be introduced in $\mathsf{RCA_0}$: given two functions f and g in $L_p(X)$, $f \geq g$ if and only if $\max(f, g) = f$.

- Equivalently, $f \geq g$ if and only if $\min(f, g) = g$.
- [$\mathsf{RCA_0}$]: $f \geq g$ if and only if $|f - g| = f - g$ if and only if $(f - g)^+ = (f - g)$ if and only if $(f - g)^- = 0$ since $f \geq g \leftrightarrow f - g \geq 0$.
- [$\mathsf{WWKL_0}$]: The ordering defined above coincides with the usual, pointwise ordering a.e.

It is also possible to define $f < g$ as $\neg(f \geq g)$.

2 Essentially bounded functions

To discuss products of functions it is necessary to consider a class of functions that is closed under products and powers: these are *essentially bounded* functions. They correspond to the classical space L_∞ whose elements are measurable functions such that for some $M \in \mathbb{R}$, $|f(x)| \leq M$ a.e. Because we do not deal with measurable functions, the definition will be slightly modified. Instead of the entire space $L_\infty(X)$, we are only interested in the functions in $L_p(X) \cap L_\infty(X)$ which can be characterized with the following definition:

Definition 2.1 ($\mathsf{RCA_0}$) *A function $f \in L_p(X)$ is said to be* essentially bounded *if $|f| \leq M$ for some $M \in \mathbb{R}^+$.*

We think of M above as the constant function M, and then interpret $|f| \leq M$ in the sense of the definition of the $\leq$ relation.

Another convenient characterization is this: f is essentially bounded if and only if $f = \max(\min(f, M), -M)$ for some $M \in \mathbb{R}^+$.

We will write "$f \in L_{p,\infty}(X)$" to mean "f is an essentially bounded function in $L_p(X)$."

Proposition 2.2 ($\mathsf{RCA_0}$) *If the function f is essentially bounded, and represented as $\langle f_n \rangle$, where each f_n is a simple function, then there exists a constant M with $|f_n| \leq M$ for all n.*

Proof. Let f be represented with $\langle f_n \rangle$. Let M be the bound for f. Then $f = \max(\min(f, M), -M)$, and the right-hand side of this expression is represented as $\langle \max(\min(f_n, M), -M) \rangle$. This exactly means that every $|f_n|$ is bounded by M, as required. □

The following properties also hold:

- [$\mathsf{RCA_0}$] If one representation of a function is essentially bounded, all representations are, and with the same bound, since if $f = g$, then $\max(f, M) = \max(g, M)$ and $\min(f, M) = \min(g, M)$.
- [$\mathsf{RCA_0}$] Classically, since X is a finite measure space, $L_\infty(X) \subseteq L_p(X)$ for all $p \geq 1$, and therefore $L_{p,\infty} = L_\infty$. Although in our framework $L_\infty(X)$ has no meaning, we will be able to show later, in Section 6, that $L_{p,\infty} = L_{q,\infty}$ for all $p, q \geq 1$, which essentially amounts to the same thing: every $L_{p,\infty}(X)$ consists of the classically essentially bounded functions, which the next item also confirms.
- [$\mathsf{WWKL_0}$]: The definition above is equivalent to the usual characterization of essential boundedness, since

  $$\forall n\, |f_n| \leq M \rightarrow \forall n\, |f_n(x)| \leq M \rightarrow |f(x)| \leq M$$

  for almost all x. The other direction follows from Proposition 1.3.

3 Characteristic functions and integrable sets

The definition of characteristic function found in [7] is made in $\mathsf{WWKL_0}$ because it presupposes that functions are defined a.e. We first give this definition, and then show how it can be modified to make sense in $\mathsf{RCA_0}$.

Definition 3.1 ($\mathsf{WWKL_0}$) *A function $f \in L_1(X)$ is a* characteristic function *if $f(x) \in \{0, 1\}$ a.e. A code for an* integrable set *E with respect to the measure μ on X is defined to be a corresponding characteristic function and $\mu(E) = \mu(f)$.*

In $\mathsf{WWKL_0}$, a characteristic function defined in such a way is essentially bounded: $0 \leq \chi_A \leq 1$. Because of this, $\chi_A^2 \in L_1(X)$ (this will be proved in Lemma 5.3); also, $\chi_A^2(x) = \chi_A(x)$ for every x in the domain. Moreover, if f is any function such that $f^2 = f$, then $f \geq 0$ and it follows that $(1 - f)^2 = 1 - f$, so $1 - f \geq 0$, or $f \leq 1$.

This motivates the following definition:

Definition 3.2 ($\mathsf{RCA_0}$) *A function $f \in L_1(X)$ is a* characteristic function *if $f^2 = f$.*

The definition of integrable sets remains the same as in 3.1.

Proposition 3.3 ($\mathsf{WWKL_0}$) *Definitions 3.1 and 3.2 coincide.*

Proof. One direction is the previous fact. The other follows from Proposition 1.3: $f^2 = f$ if and only if $f^2(x) = f(x)$ a.e. The latter means that there is an F_σ set $F = \cup_n C_n$ on which $f^2(x) = f(x)$, which is equivalent to saying $f(x) \in \{0, 1\}$ for all $x \in F$. □

Integrable sets are identified with their characteristic functions. When working in $\mathsf{WWKL_0}$, where integrable functions are pointwise defined a.e, given a characteristic function f, we can define membership in the corresponding set A: $x \in A \leftrightarrow f(x) = 1$.

The complement $\overline{A}$ of an integrable set A is integrable, since if f is a characteristic function of A, then a characteristic function of $\overline{A}$ is $1 - f$. Similarly, if A and B are integrable sets with characteristic functions f_1 and f_2, characteristic functions for $A \cup B$ and $A \cap B$ are respectively $\max(f_1, f_2)$ and $\min(f_1, f_2)$. It is not difficult to show that the characteristic function for $A \cap B$ can equivalently be written as $f_1 \cdot f_2$ and the characteristic function for $A \cup B$ as $f_1 + f_2 - f_1 \cdot f_2$. To show that these functions really correspond to complements, unions and intersections, $(WWKL)$ is needed.

Finite unions and intersections can be shown to be integrable in $\mathsf{RCA_0}$, but (ACA) is needed for the infinite case. If f_n is the characteristic function of A_n, then the characteristic function of $\cup_n A_n$ is $\sup_n f_n$ and the characteristic function of $\cap_n A_n$ is $\inf_n f_n$. Suprema and infima of integrable functions are themselves integrable (see [6]), and it is not difficult to show (in $\mathsf{WWKL_0}$, provided they exist) that suprema and infima of characteristic functions are characteristic functions themselves.

4 Defining power functions

A number of proofs below will use properties of derivatives of power functions. The purpose of this section is not to give detailed proofs of these facts, but only to convince the reader that in $\mathsf{RCA_0}$ power functions are well-defined and that basic laws of differentiation hold, along with some other simple results from calculus. The motivation from Bishop's work [1, 2], and, more importantly, from Schwichtenberg, who worked out most of these issues constructively in [5]. Most technical details of proofs will be omitted. Assume throughout that the underlying system is $\mathsf{RCA_0}$.

Standard calculus and analysis textbooks define power functions via exponentiation, i.e. $x^p = e^{p \ln x}$ for all $x > 0$. This is the approach that we will also take, but first we need to show that e^x and $\ln x$ (when $x > 0$) are continuous functions.

Based on Lemma II.6.5. in [7], power series give rise to continuous functions, so e^x can be defined as the sum of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Schwichtenberg (in [5]) defines $\ln x = \int_1^x \frac{1}{t}\, dt$, for $x > 0$. This integration is permitted in $\mathsf{RCA_0}$, because $\frac{1}{x}$ has a modulus of uniform continuity on all closed intervals $[a, b]$, with $a > 0$ (see [7], Lemma IV.2.6).

A function obtained via integration is continuous and differentiable (definition of derivatives will follow below), hence $\ln x$ is continuous for all $x > 0$. Furthermore, the reader can find a proof in [5] that $\ln x$ thus defined has the usual properties and that e^x and $\ln x$ are inverse to each other.

Therefore, when $x > 0$, the function $f(x) = e^{p \ln x}$ as a composition of continuous functions is itself continuous ([7], Lemma II.6.4) and we let $x^p = e^{p \ln x}$. Standard properties of power functions can be shown, for example that $x^{p+q} = x^p \cdot x^q$ and $(x^p)^q = x^{pq}$ (and in particular, $(x^p)^{1/p} = x$). Also, if $p \in \mathbb{N}$, $x^p = \underbrace{x \cdot \cdots \cdot x}_{p}$.

This definition only works for $x > 0$. However, this is easily fixed, by augmenting the code for x^p as a continuous function for all $x > 0$ (which, according to definition of continuous functions, means that there is a sequence of quintuples that codes it) with a countable number of conditions specifying that $0 \mapsto 0$. The result will be a code for another continuous function, the function x^p for all nonnegative numbers.

Another useful fact can be shown: suppose p is an arbitrary real number, hence represented as the limit of a strong Cauchy sequence. Then $x^p = \lim_n x^{p_n}$ when $x > 0$ (the fact is immediate when $x = 0$).

We use the following definition of derivative, first given by Bishop. Notice that the rate of convergence is computable.

Let f and g be continuous functions on a compact proper interval I and $\delta : \mathbb{Q} \to \mathbb{Q}$ such that for each ε there exists $\delta(\varepsilon) > 0$ with

$$|f(y) - f(x) - g(x)(y - x)| \leq \varepsilon |y - x|$$

whenever $x, y \in I$ and $|y - x| \leq \delta(\varepsilon)$. Then f is differentiable on the interior of I and $f'(x) = g(x)$ for all x in the domain.

With this definition, the following can be proved:

Proposition 4.1 *1.* $(f_1 + f_2)' = f_1' + f_2'$.

2. $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$.

3. $\frac{dx}{dx} = 1$.

4. $\frac{dc}{dx} = 0$.

5. $(g \circ f)' = (g' \circ f) f'$.

The proof of this proposition can be found in [2] and is constructive, applying almost word for word in RCA_0.

The following statements about power functions will be used in Section 6.

Lemma 4.2 (RCA_0) *1. The inequality* $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ *holds for all nonnegative real numbers* a *and* b. *(6.1)*

2. The inequality $|x - y|^p \leq |x^p - y^p|$ *holds for all* x *and* y. *(6.2)*

3. *The inequality* $|x^p - y^p| \le p\,|(x-y)(x^{p-1} + y^{p-1})|$ *is true for all real* x *and* y. *(6.2)*

4. *For all* $x, y \in \mathbb{R}^+$, *and* $q > p$, $y^{q/p} \ge \frac{q}{p}\, x^{\frac{q-p}{p}} (y - x) + x^{q/p}$. *(6.3)*

To prove Lemma 4.2, a number of facts are needed.

Proposition 4.3 *The function* $f(x) = x^p$ *is differentiable for all real numbers* p *and* $f'(x) = p\, x^{p-1}$.

Proof. If p is a natural number, this is proved by induction, using the product rule. The base case is item 3. in Proposition 4.1. If p is rational, the result is provided by the chain rule. Finally, if p is an arbitrary real number,

$$(x^p)' = (e^{p \ln x})' = \frac{p}{x}\, e^{p \ln x} = p\, x^{p-1},$$

by another application of the chain rule and by properties of exponents and logarithms. □

Proposition 4.4 *The function* $f(x) = x^p$ *is convex for all* $p > 1$, *that is, it is above its tangent line at every point.*

Proof. The standard proof that if $f'' > 0$, then f is convex, applies. It is easy to see that when $p > 1$, $(x^p)'' = p\,(p-1)\, x^{p-2} > 0$. □

Proposition 4.5 *The mean value theorem for* x^a*: for every* x *and* y *there is a* ξ *between them such that* $\frac{x^a - y^a}{x - y} = a\xi^{a-1}$.

Proof. The general form of the mean value theorem was proved by Hardin and Velleman in [4]. □

Proposition 4.6 *If* f *is differentiable at* x *and* $f'(x) > 0$, *then* f *is increasing in some neighborhood of* x*; if* $f'(x) < 0$, f *is decreasing; consequently, if* $f'(x_0) = 0$, $f'(x) < 0$ *(resp.* $f'(x) > 0$*) for all* $x < x_0$ *and* $f'(x) > 0$ *(resp.* $f'(x) < 0$*) for all* $x > x_0$, *then* $f(x_0)$ *is the absolute minimum (resp. absolute maximum) of* f.

Proof. If $f'(x) > 0$, then based on properties of limits and continuous functions, there is some neighborhood of x for which $\frac{f(x) - f(y)}{x - y} > 0$. This means that for $y < x$ in that neighborhood, $f(x) > f(y)$ and for $y > x$, $f(x) < f(y)$ as required. Other facts are shown similarly. □

With these facts, the proof of Proposition 4.2 is standard and omitted here (but can be found in [6]).

5 Products and Powers in L_p Spaces

Giving a meaningful, viable definition of products and powers of elements of $L_p(X)$ may seem like an innocuous problem at first, but many difficulties surround it, as we will show below. Even in classical measure theory, a product of integrable functions is not necessarily integrable, so we have to be very careful in finding the proper characterization of this concept, that will be natural in second-order arithmetic, and at the same time capture classical properties of products.

The goal is to develop as much theory as possible in RCA_0, yet this system, for the most part, cannot reason about pointwise properties of functions. Another problem is that a natural definition of products can be sensitive to representations. It is possible to construct a function f represented with the sequence of simple functions $\langle f_n \rangle$ such that $f = 0$, but $\langle f_n^2 \rangle$, though pointwise convergent to 0, does not converge in norm. Such a function can, for example, be defined in the following way:

$$f_n(x) = \begin{cases} 2n^3 x & 0 \le x \le \frac{1}{2n^2} \\ 2n - 2n^3 x & \frac{1}{2n^2} < x \le \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

A simple computation shows that $\langle f_n \rangle$ is a Cauchy sequence in $L_1([0,1])$ (converging to the zero function), while $\langle f_n^2 \rangle$ is not.

This means that if one tries to define the product of two functions by the products of their approximations, it can happen that $f_1 = f_2$ and $g_1 = g_2$, but $f_1 g_1 \neq f_2 g_2$. We want to structure the definition of products so that fg exists if it exists for some representation, and coincides with the pointwise product (0 in this case).

It makes sense to start from the pointwise product of two functions. Classically, if f, g, and h are elements of $L_p(X)$, h is said to be the product of f and g if $h(x) = f(x)g(x)$ a.e. This definition is meaningful in our framework, and it can be proved in WWKL_0 that the product of two functions, if it exists, is unique and does not depend on representations. We will call this the *pointwise* product of f and g. The shortcoming of this definition is that it uses pointwise properties of functions. However, it cannot be modified to work in RCA_0. The logical rewording would be this:

> If f, g and h are elements of $L_p(X)$, we say that $h = fg$ if there exist some representations $\langle f_n \rangle$, $\langle g_n \rangle$ and $\langle h_n \rangle$ of f, g and h respectively, such that $h(x) = f(x)g(x)$ whenever $f(x) = \lim_n f_n(x)$, $g(x) = \lim_n g_n(x)$ and $h(x) = \lim h_n(x)$ all exist.

The problem with this definition is that it is too restrictive. Because it can happen, for example, that $h = h'$ and both $h(x)$ and $h'(x)$ are defined, but $h(x) \neq h'(x)$, it is not possible to show that if $h = fg$ and $h' = h$ then $h' = fg$.

We consider instead the following, stronger characterization.

Definition 5.1 ($\mathsf{RCA_0}$) *If f, g and h are elements of $L_p(X)$ and if there exist representations $\langle f_n \rangle$ of f and $\langle g_n \rangle$ of g such that $\|h - f_n g_n\|_p \to 0$ with a fixed rate of convergence for some $h \in L_p(X)$, then we say that h is the* strong product *of f and g.*

Notice that if for some representations $\langle f_n \rangle$ and $\langle g_n \rangle$ of f and g the sequence $\langle f_n g_n \rangle$ is Cauchy with a fixed rate of convergence, then the strong product fg exists.

Proposition 5.2 ($\mathsf{WWKL_0}$) *If $fg = h$ in the strong sense, then $fg = h$ pointwise.*

Proof. Assume $\langle f_n g_n \rangle$ converges to h in L_p norm, and let x be such that $f(x), g(x)$ and $h(x)$ are defined. Then, since $\|h - f_n g_n\|_p \to 0$ with a fixed rate of convergence, one can construct subsequences $\langle f_n' \rangle$ and $\langle g_n' \rangle$ of $\langle f_n \rangle$ and $\langle g_n \rangle$ such that $\|h - f_n' g_n'\|_p \leq 2^{-n}$ for all n.

Let f' be the function given by $\langle f_n' \rangle$, let g' be the function given by $\langle g_n' \rangle$, and let h' be the function represented as $\langle f_n' g_n' \rangle$. Then $f' = f$, $g' = g$, and $h' = h$; furthermore, $f'(x)$ and $g'(x)$ are defined because $\langle f_n' \rangle$ is a subsequence of $\langle f_n \rangle$, and similarly for $\langle g_n' \rangle$ and $\langle g_n \rangle$. Therefore, $f'(x) = f(x)$, and $g'(x) = g(x)$ implying

$$f(x)g(x) = f'(x)g'(x) = (\lim_n f_n'(x))(\lim_n g_n'(x)) = \lim_n f_n'(x)g_n'(x) = \lim_n h_n'(x).$$

Because $\lim_n h_n'(x)$ exists, it is in fact equal to $h(x)$, consequently, $f(x)g(x) = h(x)$ as required. □

This proof almost went through in $\mathsf{RCA_0}$. The only argument that required weak-weak König's lemma was in stating that $h(x) = h'(x)$.

From now on, we are going to use Definition 5.1: the remainder of the section will show that this is justified, at least in the case when one of the functions is essentially bounded. The strong product does not depend on representations of f and g. Showing that the product, if it exists, is unique, requires $\mathsf{WWKL_0}$. In this case, the proof is easy, since if $h_1 = fg$ and $h_2 = fg$, then $h_1(x) = f(x)g(x) = h_2(x)$ a.e. which implies that $h_1 = h_2$. Notice that the definition naturally takes care of our counterexample (1) above, as for the representation of the zero function $\langle 0 \rangle$, its product with itself is also the zero function, as required.

The main result of this section will be Proposition 5.5, which states that the strong product of two functions, one of which is essentially bounded, exists. First we show a weaker result.

Proposition 5.3 ($\mathsf{RCA_0}$) *If $f, g \in L_{p,\infty}$, then the strong product fg exists and is also in $L_{p,\infty}$. In fact, for any choice of representations of f and g the sequence $\langle f_n g_n \rangle$ converges, and to the same function.*

Proof. Let f be represented with $\langle f_n \rangle$, such that for all n, $|f_n| \leq M_f$ and g be represented with $\langle g_n \rangle$, with $|g_n| \leq M_g$ for all n. To show that the sequence

$\langle f_n g_n \rangle$ is convergent, we show it is Cauchy with a fixed rate of convergence. Let $m < n$.

$$\begin{aligned} \|f_n g_n - f_m g_m\|_p &= \|f_n g_n - f_n g_m + f_n g_m - f_m g_m\|_p \\ &\leq \|f_n(g_n - g_m)\|_p + \|g_m(f_n - f_m)\|_p \\ &\leq M_f 2^{-m} + M_g 2^{-m}. \end{aligned}$$

Pulling out constants outside of the norm is justified, because all functions in question are simple, and measure is monotonic on simple functions.

Showing that the product does not depend on the representations of f and g is analogous. Let $\langle f_n \rangle$ and $\langle f'_n \rangle$ represent f, and $\langle g_n \rangle$ and $\langle g'_n \rangle$ represent g. This means that for all n, $\|f_n - f'_n\|_p \leq 2^{-n+1}$ and $\|g_n - g'_n\|_p \leq 2^{-n+1}$. By assumption, $\|h - f_n g_n\|_p \to 0$ with a fixed rate of convergence. Recall also that all representations of an essentially bounded function have the same bound.

$$\begin{aligned} \|h - f'_n g'_n\|_p &\leq \|h - f_n g_n\|_p + \|f_n g_n - f'_n g_n\|_p + \|f'_n g_n - f'_n g'_n\|_p \\ &\leq \|h - f_n g_n\|_p + M_g 2^{-n+1} + M_f 2^{-n+1}, \end{aligned}$$

which implies that $\|h - f'_n g'_n\|_p \to 0$ also (with a fixed rate of convergence). It is immediate that $|fg| \leq M_f M_g$. □

To prove the more general claim, we will need the following (stronger) condition $(\star)$:

> Functions f and g are elements of $L_p(X)$, represented, respectively as $\langle f_n \rangle$ and $\langle g_n \rangle$ and there exist a sequence of functions $\langle h_n \rangle$ as well as a function h in $L_p(X)$ such that $f_n g_m \to h_n$ when $m \to \infty$ and $h_n \to h$ when $n \to \infty$.

To be able to use this characterization, we need to show that it implies Definition 5.1.

Proposition 5.4 (RCA_0) *If two functions f and g in $L_p(X)$ satisfy property $(\star)$, then fg exists in the sense of Definition 5.1.*

Proof. Assume $f_n g_m \xrightarrow{m} h_n \xrightarrow{n} h$. For each n let m_n be such that $\|h_n - f_n g_{m_n}\|_p \leq 2^{-n}$. Then it is clear that $\langle f_n \rangle$ represents f and $\langle g_{m_n} \rangle$ represents g and $\|h - f_n g_{m_n}\|_p \xrightarrow{n} 0$. □

If another assumption is added, namely that one of the functions is essentially bounded, then the other direction holds true as well. A sketch of the proof is as follows.

Suppose $\|h - f_n g_n\|_p \to 0$ for some representations of f and g, and that f is essentially bounded. According to the first fact below, for a fixed n, there always exists h_n such that $\|h_n - f_n g_m\|_p \xrightarrow{m} 0$. Then

$$\begin{aligned} \|h - h_n\|_p &\leq \|h - f_n g_n\|_p + \|f_n g_n - f_n g_m\|_p + \|f_n g_m - h_n\|_p \\ &\leq \|h - f_n g_n\|_p + M_f 2^{-n} + \|f_n g_m - h_n\|_p, \end{aligned}$$

which, since m is arbitrary, implies that $\|h - h_n\|_p \leq \|h - f_n g_n\|_p + M_f 2^{-n}$. Letting $n \to \infty$, it follows that $h_n \to h$ in $L_p(X)$. This still does not conclude the proof, because the conclusion should hold for any representation of f and g. Lemma 5.5 will show that this is indeed the case.

Observe now some facts about products implied by ($\star$):

1. [$\mathsf{RCA_0}$]: The sequence of functions $\langle h_n \rangle$ in the definition above always exists, since $\|f_n g_m - f_n g_k\|_p \leq M_{f_n} \|g_m - g_k\|_p$, where $|f_n| \leq M_{f_n}$, so for a fixed n the sequence $\langle f_n g_m \rangle$ is strong Cauchy.

2. [$\mathsf{RCA_0}$]: The definition is independent of the order in which the limits are taken: if $f_n g_m \xrightarrow{n} h_m \xrightarrow{m} h$ and $f_n g_m \xrightarrow{m} w_n \xrightarrow{n} w$, then $h = w$. Furthermore, one limit exists if and only if the other does.

   This is because we have $\|h - w\|_p \leq \|h - h_m\|_p + \|h_m - f_n g_m\|_p + \|f_n g_m - w_n\|_p + \|w_n - w\|_p$ for all n and m. For every k it is possible to choose m and n large enough such that $\|h - w\|_p \leq 2^{-k}$.

   The second part is proved by a slight modification of the argument, since for example $\|h - w_n\|_p \leq \|h - h_m\|_p + \|h_m - f_n g_m\|_p + \|f_n g_m - w_n\|_p$ for all m (by the previous fact the sequence $\langle h_n \rangle$ always exists).

We now come to the promised main result of this section, which will, among other things, enable us to integrate over arbitrary integrable sets. Notice that ($\star$) is precisely the characterization needed to prove the claim.

Proposition 5.5 ($\mathsf{RCA_0}$) *Let f and g be in $L_p(X)$ for some p. If g is essentially bounded, then fg exists as an element of $L_p(X)$. Furthermore, all representations of f and g yield the same product.*

Proof. Let f be represented with $\langle f_n \rangle$, and g with $\langle g_n \rangle$. Assume that g is essentially bounded with $|g| \leq M_g$.

Fix n. Since $f_n \in S(X)$, it is bounded, and there exists M_{f_n} such that $|f_n(x)| \leq M_{f_n}$ for all x. Consider $\langle f_n g_k \mid k \in \mathbb{N} \rangle$. Let $m > k$.

$$\|f_n g_k - f_n g_m\|_p \leq M_{f_n} 2^{-k}.$$

Therefore, $\langle f_n g_k \mid k \in \mathbb{N} \rangle$ is a Cauchy sequence that represents $f_n g$.

Next show that the sequence $\langle f_n g \mid n \in \mathbb{N} \rangle$ is strong Cauchy in $L_p(X)$. For $n > m$

$$\begin{aligned} \|f_m g - f_n g\|_p &= \|(f_m - f_n)\, g\|_p \\ &\leq M_g \|f_m - f_n\|_p \\ &\leq M_g 2^{-m}. \end{aligned}$$

The first inequality would be almost trivially true in $\mathsf{WWKL_0}$. In $\mathsf{RCA_0}$, however, it requires some work. It follows from these two facts:

$$\|(f_m - f_n)\, g_k\|_p \leq M_g \|f_m - f_n\|_p,$$

which is true by monotonicity of measure for simple functions, and

$$\|(f_m - f_n)\, g_k\|_p \to \|(f_m - f_n)\, g\|_p. \tag{2}$$

That this is enough follows from the more general fact (with a simple proof) that, if $\langle x_n \rangle$ is a sequence of real numbers that converges to x, and if y is such that for all n, $x_n \leq y$, then $x \leq y$ as well. It remains to show (2).

It is true that $\|g - g_k\|_p \leq 2^{-k}$. Furthermore,

$$\begin{aligned} \|(f_m - f_n)\, g - (f_m - f_n)\, g_k\|_p &= \|(g - g_k)(f_n - f_m)\|_p \\ &\leq (M_{f_n} + M_{f_m})\, 2^{-k}, \end{aligned}$$

since f_n and f_m are simple functions. This means that $(f_m - f_n)\, g_k \to (f_m - f_n)\, g$ in $L_p(X)$ for each fixed m and n when $k \to \infty$, which also concludes the proof of the above claim.

Consequently, there exists $h \in L_p(X)$ such that $\|h - f_n g\|_p \to 0$ as $n \to \infty$. By property ($\star$), $h = fg$.

To prove the second part, let f and g be as above, such that $f_n g_m \xrightarrow{m} h_n \xrightarrow{n} h$ for some representations of f and g (recall that, if one of the functions is essentially bounded, then property ($\star$) coincides with the definition of products). We will show that if $\langle f'_n \rangle$ and $\langle g'_n \rangle$ are different representations of f and g, there exists a sequence of functions $\langle h'_n \rangle$ in $L_p(X)$ such that $f'_n g'_m \xrightarrow{m} h'_n \xrightarrow{n} h$.

Recall that for each n, h'_n exists. Since each f'_n is a simple function, for every n there exists a positive constant $M_{f'_n}$ such that $|f'_n| \leq M_{f'_n}$. Next,

$$\begin{aligned} \|h_n - h'_n\|_p &\leq \|h_n - f_n g_m\|_p + \|f_n g_m - f'_n g_m\|_p \\ &+ \|f'_n g_m - f'_n g'_m\|_p + \|f'_n g'_m - h'_n\|_p \\ &\leq \|h_n - f_n g_m\|_p + M_g 2^{-n+1} \\ &+ M_{f_n} 2^{-m+1} + \|f'_n g'_m - h'_n\|_p. \end{aligned}$$

Since m is arbitrary, it follows that $\|h_n - h'_n\|_p \leq M_g 2^{-n+1}$. Furthermore,

$$\|h - h'_n\|_p \leq \|h - h_n\|_p + \|h_n - h'_n\|_p \leq \|h - h_n\|_p + M_g 2^{-n+1},$$

which approaches 0 when $n \to \infty$ and with a fixed rate of convergence. □

Since $g = f\chi_G$ is integrable by Proposition 5.5, it is valid to define $\int_G f = \mu(f\chi_G)$. We are now able to integrate over sets.

Similarly, to define a function by cases, for f to be f_k on M_k, where $f_k \in L_1(X)$ and M_k a integrable set, define

$$f = \sum_{k=1}^{n} f_k \chi_{M_k}.$$

Linear combinations of integrable functions are integrable, and since $f_k \chi_{M_k}$ is integrable for each k, f is integrable.

Corollary 5.6 (WWKL_0) *If f or g is essentially bounded, and if $fg = h$ pointwise a.e., then $fg = h$ in the strong sense.*

Proof. Since the strong product in this case exists, and is unique, the two have to coincide. □

To summarize:
In RCA_0 we have the following:

- The strong product, if it exists, does not depend on representations.
- If one of the functions is essentially bounded, the strong product exists and is unique.

In WWKL_0 the following hold:

- The strong product, if it exists, is unique. representations.
- The pointwise product, if it exists, is unique and does not depend on representations.
- If h is the strong product of f and g, then $h = fg$ a.e.
- If one of the functions is essentially bounded, then their pointwise product exists.
- If one of the functions is essentially bounded, and $h = fg$ a.e, then $h = fg$ in the strong sense.

Thus, when f or g is essentially bounded, Definition 5.1 provides a characterization of the pointwise product that is useful in the absence of WWKL_0, but provably equivalent to the pointwise definition in the presence of WWKL_0. This situation is satisfactory. In general, however, the two may not be equivalent for more general functions; in other words, when neither f nor g is essentially bounded, they may have a pointwise product that is not a product in the strong sense. This question is still open.

Defining powers of functions is also potentially problematic. (Before continuing, note that the discussion below makes sense only for nonnegative functions.)
As with products, we can give two definitions. The first is pointwise.

Definition 5.7 (WWKL_0) *Let $f \in L_p(x)$ and let $k \in \mathbb{R}$. If there exists a representation $\langle f_n \rangle$ of f and $h \in L_p(X)$ such that $f^k(x) = h(x)$ a.e., we say that h is the* pointwise k^{th} power *of f.*

The other alternative is to consider convergence in norm.

Definition 5.8 (RCA_0) *Let $f \in L_p(X)$ and let $k \in \mathbb{R}$. If there exists a representation $\langle f_n \rangle$ of f such that $\|h - f_n^k\|_p \to 0$ for some $h \in L_p(X)$, we say that h is the* strong k^{th} power *of f.*

The relationships between the two definitions in RCA_0 and WWKL_0 are the same as the ones for products.

Definition 5.8 is closely related to Definition 5.1, but is weaker. Although we would hope that the two would coincide when considering f^2 for some function f, it is not necessarily the case: if f^2 exists in the sense just defined, then $f \cdot f$ exists as well. The reverse is still an open question: it is conceivable that two different representations of f are taken to form the product, whereas for no representation is $\langle f_n^2 \rangle$ convergent. As before, if f is an element of $L_{p,\infty}$, the definitions do coincide. Assuming $\langle f_n \rangle$, $\langle f'_n \rangle$ and $\langle f''_n \rangle$ are all representations of an essentially bounded function with bound M_f, and there is some function h such that $\|h - f'_n f''_n\|_p \to 0$, the conclusion follows from the fact that

$$\begin{aligned} \|h - f_n^2\|_p &\leq \|h - f'_n f''_n\|_p + \|f'_n f''_n - f'_n f_n\|_p + \|f'_n f_n - f_n^2\|_p \\ &\leq \|h - f'_n f''_n\|_p + M_f 2^{-n+1} + M_f 2^{-n+1}. \end{aligned}$$

For this reason, as in the case of products, it would be reasonable to restrict the definition to essentially bounded functions only.

6 Further Properties of L_p Spaces

The next task is to determine which of the standard properties of L_p spaces carry over to our framework, and for those that do, whether the same proofs can be used and in which subsystem.

First we prove Hölder's inequality.

Lemma 6.1 (RCA_0) *If $f \in L_p(X)$ and $g \in L_q(X)$ (where $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$) are given with representations $\langle f_n \rangle$ and $\langle g_n \rangle$, then the sequence $\langle f_n g_n \rangle$ is strong Cauchy in $L_1(X)$, therefore fg is integrable and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Proof. In the standard proof, integrability of fg is not established separately, as it is a consequence of Hölder's inequality: $\int fg < \infty$, which is enough in the classical setup. We cannot follow this reasoning, since fg cannot be integrated without the knowledge that it is integrable in the sense of one of the definitions of products.

We are able to show, however, that for every n

$$\|f_n g_n\|_1 \leq \|f_n\|_p \, \|g_n\|_q, \tag{3}$$

using the standard argument. The inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ holds for all non-negative real numbers a and b is provable in RCA_0 (shown in Section 4). Let u and v be any two simple functions. Let $a = \frac{|u(x)|}{\|u\|_p}$ and $b = \frac{|v(x)|}{\|v\|_q}$ in the above inequality, obtaining

$$\frac{|u(x)|}{\|u\|_p} \cdot \frac{|v(x)|}{\|v\|_q} \leq \frac{1}{p} \frac{|u(x)|^p}{\|u\|_p^p} + \frac{1}{q} \frac{|v(x)|^q}{\|v\|_q^q}.$$

Integrating yields the desired inequality. In particular, when $u = f_n$ and $v = g_n$, (3) follows.

Next, show that fg exists (and is integrable). Let $m < n$. Since $f_n - f_m$ and $g_n - g_m$ are simple, Hölder's inequality for simple functions applies.

$$\begin{aligned} \|f_n g_n - f_m g_m\|_1 &= \|f_n g_n - f_n g_m + f_n g_m - f_m g_m\|_1 \\ &\leq \|f_n(g_n - g_m)\|_1 + \|g_m(f_n - f_m)\|_1 \\ &\leq \|f_n\|_p \|g_n - g_m\|_q + \|g_m\|_q \|f_n - f_m\|_p. \end{aligned}$$

Every Cauchy sequence of real numbers is bounded: $\|f_n\|_p \leq \|f_1\|_p + 2^{-1} = M_1$ and $\|g_n\|_1 \leq \|g_1\| + 2^{-1} = M_2$. Then

$$\|f_n g_n - f_m g_m\|_1 \leq M_1 2^{-m} + M_2 2^{-m},$$

which is a Cauchy sequence with a fixed rate of convergence. Since $\langle f_n g_n \rangle$ is a strong Cauchy sequence, the strong product fg exists, and taking the limit in (3) is justified. The Hölder's inequality follows. □

It is worth pointing out that our definition of L_p spaces is different from that of most authors. Typically, a function is defined to be in $L_p(X)$ if and only if f is measurable and $|f|^p \in L_1(X)$. We can prove that our definition is equivalent to a similar characterization. Since we cannot discuss arbitrary functions, the following lemma has to be stated for sequences instead.

Lemma 6.2 (RCA_0) *The sequence $\langle f_n \rangle$ of simple functions is an element of $L_p(X)$ if and only if $\langle (f_n^+)^p \rangle$ and $\langle (f_n^-)^p \rangle$ are elements of $L_1(X)$.*

Proof. Let $\langle f_n \rangle$ be given and assume $\langle (f^{\pm})^p \rangle \in L_1(X)$. The inequality $|x-y|^p \leq |x^p - y^p|$ holds for all $x, y \in \mathbb{R}$ (see Section 4) so

$$|f_n^{\pm} - f_m^{\pm}|^p \leq |(f_n^{\pm})^p - (f_m^{\pm})^p|$$

for all n, m. Applying μ to the inequality yields

$$\|f_n^{\pm} - f_m^{\pm}\|_p^p \leq \|(f_n^{\pm})^p - (f_m^{\pm})^p\|_1 \leq 2^{-m}$$

when $m < n$. Therefore, $\langle f_n^+ \rangle$ and $\langle f_n^- \rangle$ are Cauchy in $L_p(X)$, meaning that f^+ and f^- are elements of $L_p(X)$; consequently $\langle f_n \rangle$ defines a function $f \in L_p(X)$, since $f = f^+ - f^- \in L_p(X)$.

For the other direction, assume $\langle f_n \rangle$ represents a function $f \in L_p(X)$. It suffices to show that if g is nonnegative and $g \in L_p(X)$, then $g^p \in L_1(X)$ (since f^+ and f^- are nonnegative and both are elements of $L_p(X)$).

Based on the inequality $|x^p - y^p| \leq p\,|(x-y)(x^{p-1} + y^{p-1})|$, which is true for all real x and y (see Section 4), it follows that

$$|g_n^p - g_m^p| \leq p\,|(g_n - g_m)(g_n^{p-1} + g_m^{p-1})|,$$

for all n and m. Assume $m < n$. After integrating, and with the aid of Hölder's inequality,

$$\begin{aligned} \|g_n^p - g_m^p\|_1 &\leq p\,\|g_n - g_m\|_p\,\|g_n^{p-1} + g_m^{p-1}\|_q \\ &\leq p\,\|g_n - g_m\|_p\,(\|g_n^{p-1}\|_q + \|g_m^{p-1}\|_q), \end{aligned}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Because $g \in L_p(X)$, $\|g_n - g_m\|_p \leq 2^{-m}$. It remains to estimate $\|g_n^{p-1}\|_q$ and $\|g_m^{p-1}\|_q$. Notice that $p = (p-1)q$.

$$\|g_n^{p-1}\|_q = [\mu(g_n^{(p-1)q})]^{1/q} = [\mu(g_n^p)]^{1/q} = \|g_n\|_p^{p/q}.$$

The sequence $\langle \|g_n\|_p \rangle$ is Cauchy in $\mathbb{R}$ and thus bounded; let M be an upper bound. Then

$$\|g_n^p - g_m^p\|_1 \leq p\,M^{p/q}\,2^{-n+1},$$

and $\langle g_n^p \rangle$ defines an integrable function, g^p, therefore, both $(f^+)^p$ and $(f^-)^p$ are integrable. □

Next we can determine the relationship among the $L_p(X)$ spaces. The proof is adapted from [8].

Lemma 6.3 (RCA_0) *For $p < q$, $L_q(X) \subseteq L_p(X)$.*

Note: The statement $L_q(X) \subseteq L_p(X)$ is an abbreviation for "every function f that can be represented as a Cauchy sequence in the L_q norm has an equivalent representation (in the L_q sense) which is Cauchy in $L_p(X)$."

Proof. It is sufficient to show that if $\langle f_n \rangle \in L_q(X)$, then $\langle f_n \rangle \in L_p(X)$, that is, if $\|f_m - f_n\|_q \leq 2^{-m}$ for all m and n such that $m < n$, then $\|f_m - f_n\|_p \leq 2^{-m}$ as well. It is clearly enough to show that $\|g\|_p \leq \|g\|_q$ for $g \in S(X)$. The claim is derived from the following: for $x, y \in \mathbb{R}^+$, $y^{q/p} \geq \frac{q}{p}\,x^{\frac{q-p}{p}}\,(y - x) + x^{q/p}$. This inequality is provable in RCA_0 (sketch of proof in Section 4). In particular, let $f = |g|^p$ (f is in $C(X)$) and let $y = f(t)$:

$$[f(t)]^{q/p} \geq \frac{q}{p}\,x^{\frac{q-p}{p}}\,(f(t) - x) + x^{q/p}.$$

Recall that $\mu(1) = 1$. Since measure is monotonic on $C(X)$, it follows that

$$\mu(f^{q/p}) \geq \frac{q}{p}\,x^{\frac{q-p}{p}}\,(\mu(f) - x) + x^{q/p},$$

and since the above inequality is true for all $x \in \mathbb{R}$, let $x = \mu(f)$, obtaining

$$\mu(f^{q/p}) \geq (\mu(f))^{q/p},$$

which after some algebraic manipulation turns into $\|g\|_q \geq \|g\|_p$. □

Furthermore, if $p < q$ and $f \in L_q(X)$, then $\|f\|_p \leq \|f\|_q$. Lemma 6.3 also implies that every function in $L_p(X)$, for any $p > 1$, is pointwise defined a.e. (as it is also an element of $L_1(X)$).

7 Integrable Functions and Sets

We now restrict our attention to the space $L_1(X)$.

Lemma 7.1 ($\mathsf{ACA_0}$) *For $f \in L_1(X)$ and any $c \in \mathbb{R}$ the sets $\{x \mid f(x) > c\}$ and $\{x \mid f(x) \geq c\}$ are integrable.*

Proof. Set $f = \langle f_n \rangle$, with $f_n \in S(X)$. For each n and each c, $\{x \mid f_n(x) > c\}$ is open, and $f(x) = \lim_n f_n(x)$ for all x in the domain of f, a full F_σ set F. Since characteristic functions of sets that differ by a null set are equal in $L_1(X)$, there is no loss of generality in assuming that $x \in X$ instead of $x \in F$.

$$\begin{aligned} \{x \mid f(x) > c\} &= \{x \mid \exists m \, \exists k \, \forall n \geq k \; f_n(x) > c + 2^{-m}\} \\ &= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \{x \mid f_n(x) > c + 2^{-m}\}, \end{aligned}$$

which is integrable, as $\{x \mid f_n(x) > c + 2^{-m}\}$ is an open set and infinite combinations of integrable sets are integrable (in $\mathsf{ACA_0}$). The second claim follows from the first, using complements. □

It is not difficult to show that the previous claim reverses to $\mathsf{ACA_0}$.

Theorem 7.2 ($\mathsf{RCA_0}$) *The following are equivalent:*

1. *For $f \in L_1(X)$ and any $c \in \mathbb{R}$ the set $\{x \mid f(x) > c\}$ is integrable.*
2. *(ACA).*

Proof. That (1) → (2) was proved in Lemma 7.1. To prove the other direction, we will use item 2. from Lemma 1.1. For that purpose, let a sequence $\langle a_n \rangle$ of nonnegative numbers be given, such that for all n, $\sum_{i<n} a_i \leq 1$. We are going to construct a function $f \in L_1([0,1])$ such that

$$\mu(\{x \mid f(x) > 0\}) = \sum_n a_n. \tag{4}$$

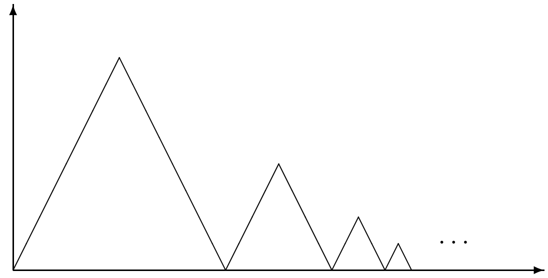
For this purpose, define a Cauchy sequence of simple functions $\langle f_n \rangle$ such that $\mu(f_n) = \sum_{i \leq n} \frac{a_i}{2^i}$. This can be accomplished as follows. First define f_1:

$$f_1(x) = \begin{cases} \frac{2}{a_1}x & 0 \leq x < \frac{a_1}{2} \\ 1 - \frac{2}{a_1}x & \frac{a_1}{2} \leq x < a_1 \\ 0 & a_1 \leq x. \end{cases}$$

Clearly, $\mu(f_1) = a_1$. Similarly, assuming f_i is defined for $i < n$, assuming it satisfies (4), define

$$f_n(x) = \begin{cases} f_i(x) & \sum_{k \leq i-1} a_k \leq x < \sum_{k \leq i} a_i, \; i < n, \\ \frac{2^n}{a_n} x, & \sum_{k \leq n-1} a_k \leq x < \sum_{k \leq n-1} a_k + \frac{a_n}{2}, \\ \frac{1}{2^{n-1}} - \frac{2^n}{a_n} x, & \sum_{k \leq n-1} a_k + \frac{a_n}{2} \leq x < \sum_{k \leq n} a_k, \\ 0, & \sum_{k \leq n} a_k \leq x. \end{cases}$$

Graphically:



where the length of the side of the i^{th} triangle lying on the x-axis is a_i and each triangle has area $\frac{a_i}{2^i}$, hence $\mu(f_n) = \sum_{i \leq n} \frac{a_i}{2^i}$. The function corresponding to this graph is a simple function.

First let us show that $\langle f_n \rangle$ is a strong Cauchy sequence. Notice that the sequence $\langle f_n \rangle$ is increasing, so if $m < n$,

$$\begin{aligned} \|f_n - f_m\| &= \mu(f_n - f_m) = \mu(f_n) - \mu(f_m) \\ &= \sum_{k=m+1}^{n} \frac{a_k}{2^k} \leq \sum_{k=m+1}^{n} \frac{1}{2^k} \leq \frac{1}{2^{m+1}}, \end{aligned}$$

as required. This means that there is a function $f \in L_1([0,1])$ represented with this sequence. Furthermore, the set $M = \{x \mid f(x) > 0\}$ is precisely the domain of f with the exception of countably many points of the form $x = \sum_{i \leq n} a_i$, where $f(x) = 0$, but countable sets don't affect measure. Since M is integrable, $\mu(M)$ exists, and it is not hard to show that $\mu(M) = \sum_n a_n$, which completes the proof. □

We now adapt a standard result from measure theory. The proof is similar to the standard one.

Lemma 7.3 (ACA_0) *Let $\langle f_n \rangle$ be a pointwise convergent sequence of functions in $L_1(X)$, which converges in norm to an integrable function f. Then $\langle f_n \rangle$ converges to f pointwise a.e.*

Proof. First show that $\langle f_n \rangle$ has a subsequence that converges pointwise to f. Since $\|f - f_n\| \to 0$, for each k let f_{n_k} be such that $\|f - f_{n_k}\| \leq 2^{-k}$.

We are going to show that $\mu(\{x \mid \lim_k f_{n_k}(x) \neq f(x)\}) = 0$, which will, by regularity of measure for G_δ sets, imply that $\lim_k f_{n_k}(x) = f(x)$ a.e.

Let $U = \{x \mid \lim_k f_{n_k}(x) \neq f(x)\}$. Since

$$\lim_k f_{n_k}(x) \neq f(x) \leftrightarrow \exists m \, \forall l \, \exists k \geq l \, (|f(x) - f_{n_k}(x)| \geq 2^{-m}),$$

the set U can be represented as

$$U = \bigcup_m \bigcap_l \bigcup_{k \geq l} U_{mk},$$

where $U_{mk} = \{x \mid |f(x) - f_{n_k}(x)| \geq 2^{-m}\}$.

On the one hand, $\int_{U_{mk}} |f_{n_k} - f| \leq \|f_{n_k} - f\| \leq 2^{-k}$. On the other, $\int_{U_{mk}} |f_{n_k} - f| \geq \int_{U_{mk}} 2^{-m} = 2^{-m}\mu(U_{mk})$. Therefore, $\mu(U_{mk}) \leq 2^{m-k}$.

Next, $\mu(\bigcup_{k \geq l} U_{mk}) \leq \sum_{k \geq l} 2^{m-k} = 2^{m-l+1}$, which implies

$$\mu(\bigcap_l \bigcup_{k \geq l} U_{mk}) \leq 2^{m-l+1}$$

for all l, and hence this set has measure 0. The set U, consequently, is the union of sets of measure 0, so itself has measure 0. As was mentioned earlier, since it is simple enough in structure, U is contained in a null G_δ set, so it can be assumed that U is a G_δ set. Hence, the subsequence $\langle f_{n_k} \rangle$ converges a.e. to f.

This concludes the first part of the proof. It remains to show that the entire sequence converges pointwise to f. By pointwise convergence of $\langle f_n \rangle$, for all x outside of a null G_δ set there exists some $\hat{x}$ such that $f_n(x) \to \hat{x}$. It remains to show that $\hat{x} = f(x)$ a.e. Since

$$|\hat{x} - f(x)| \leq |\hat{x} - f_{n_k}(x)| + |f_{n_k}(x) - f(x)|,$$

it is easy to see that this is true. □

Compare this lemma to the result from Section 5 that if $\langle f_n g_n \rangle$ converges strongly to h, then it converges pointwise to h as well. The proof of this fact only needed $\mathsf{WWKL_0}$, whereas Lemma 7.3 is proved in $\mathsf{ACA_0}$. There is no contradiction between the two. The functions in Section 5 were simple, and the convergence was strong: we have neither of the two in the previous lemma.

With additional assumptions, the reverse also holds, i.e. pointwise convergence can imply convergence in norm.

Lemma 7.4 ($\mathsf{ACA_0}$) *If $\langle f_n \rangle$ is a sequence of integrable functions that converges pointwise to an integrable function f, and if in addition $\|f_n\| \leq \|f\|$ (or $\|f_n\| \to \|f\|$) then $f_n \to f$ in $L_1(X)$.*

A proof is given in the next section.

8 The Monotone Convergence Theorem

In [10], Yu showed that a weaker version of the monotone convergence theorem is equivalent to $(WWKL)$ over $\mathsf{RCA_0}$: if a monotonic sequence of integrable functions pointwise converges to a known integrable function, then this sequence is also convergent in L_1 norm. If the pointwise limit is not known beforehand to be an integrable function, (ACA) is needed in the proof.

Theorem 8.1 (ACA_0) *Assume $\langle f_n \rangle$ is a monotonic sequence of functions in $L_1(X)$ with bounded measure. Then there is $f \in L_1(X)$ such that $\|f_n - f\| \to 0$, and $\mu(f_n) \to \mu(f)$.*

Proof. Without loss of generality, let $\langle f_n \rangle$ be increasing, and $\mu(f_n) \le M$ for all n. Since $\langle f_n \rangle$ is increasing, the sequence $\langle \mu(f_n) \rangle$ is an increasing sequence of real numbers (Proposition 1.3) and as it is bounded, it is convergent and therefore Cauchy so

$$\forall \varepsilon\, \exists N\, \forall n > m > N\ (|\mu(f_n) - \mu(f_m)| < \varepsilon)$$

and

$$|\mu(f_n) - \mu(f_m)| = \mu(f_n) - \mu(f_m) = \mu(f_n - f_m) = \mu(|f_n - f_m|),$$

which means that

$$\forall \varepsilon\, \exists N\, \forall n > m > N\ (\|f_n - f_m\| \le \varepsilon).$$

The sequence $\langle f_n \rangle$ is Cauchy, hence convergent in ACA_0. Because $L_1(X)$ is complete, the sequence converges to an integrable function f such that $\mu(f_n) \to \mu(f)$. □

For the reversal, it will be necessary to show that a function of the form $\sum_k c_k \chi_{I_k}$ (where I_k are disjoint, half-open intervals that cover $[0,1]$) is integrable. The following situation arises:

Lemma 8.2 (RCA_0) *Let $0 = a_0 < a_1 < \cdots \le 1$, define $I_k = [a_k, a_{k+1})$ and $\cup_k I_k = [0,1]$. Then $\chi_{I_k} \in L_1([0,1])$ and if $\langle c_k \rangle$ is a sequence of rational numbers with $|c_{k+1} - c_k| \le M$ for all n and some constant M, then $\sum_n c_k \chi_{I_k} \in L_1(x)$.*

Proof. It is necessary to find a Cauchy sequence (with a fixed rate of convergence) $\langle g_n \rangle$ of functions in $C(X)$ such that $\lim_n g_n = \sum_k c_k \chi_{I_k}$. Let $l_k = a_{k+1} - a_k$. Fix n and define

$$g_n(x) = c_k, \quad \text{for } a_k + l_k \cdot 2^{-k-n} \le x \le a_{k+1} - l_k \cdot 2^{-k-n},$$

(except when $k = 0$, in which case the condition is $0 \le x \le a_1 - l_0 \cdot 2^{-n}$) and make it linear otherwise. Let $m < n$. Then g_n and g_m differ only on the open interval

$$(a_{k+1} - l_k \cdot 2^{-k-m}, a_{k+1} + l_{k+1} \cdot 2^{-k-1-m})$$

for each k. Each of these intervals has length $l_{k+1} \cdot 2^{-k-1-m} + l_k \cdot 2^{-k-m}$, which is less than 2^{-k-m-1}, and $|g_n - g_m| \le M$ on each, hence

$$\mu(|g_n - g_m|) \le \sum_k M 2^{-k-m-1} = M 2^{-m+2}.$$

The sequence $\langle g_n \rangle$ therefore represents an integrable function: the function $\sum_k c_k \chi_{I_k}$. □

The reversal can now be proved.

Theorem 8.3 ($\mathsf{RCA_0}$) *The following are equivalent:*

1. *(ACA).*
2. *Monotone convergence theorem for an arbitrary measure on a complete separable metric space.*
3. *Monotone convergence theorem for the Lebesgue measure on* $[0,1]$.

Proof. $(1) \to (2)$ was proved above and $(2) \to (3)$ is immediate. It remains to show $(3) \to (1)$. Let $\langle a_n \rangle$ be a monotonic bounded sequence of real numbers. We may assume $0 \le a_1 \le a_2 \le \cdots \le 1$. The goal is to show that this sequence is convergent, which will in turn imply arithmetic comprehension (by Lemma 1.1). We will construct an increasing, pointwise convergent sequence of functions $\langle f_n \rangle$ in $L_1[0,1]$, such that $\mu(f_n) = a_n$. Assume the monotone convergence theorem holds. One of the conclusions of the theorem was that $\mu(f_n)$ converges. This will imply that $\langle a_n \rangle$ is convergent as well.

It remains to construct the desired sequence of functions.

$$f_n(x) = \begin{cases} a_1 & x \le \frac{1}{2} \\ 2a_2 - a_1 & \frac{1}{2} < x \le \frac{3}{4} \\ \dots & \\ 2^{n-1}a_n - 2^{n-2}a_{n-1} - \cdots - a_1 & \frac{2^{n-1}-1}{2^{n-1}} < x \le 1. \end{cases}$$

For each fixed n, f_n is a step function that jumps at every $\frac{2^k-1}{2^k}$ for $1 \le k \le n-1$, with step size bounded by 2^{n-1}. According to Lemma 8.2, $f_n \in L_1([0,1])$ for each n. Furthermore, $\mu(f_n) = a_n$, the sequence is increasing and converges everywhere except possibly at 1, and is therefore as required. This completes the proof. □

The monotone convergence theorem helps establish integrability of suprema and infima of sequences of integrable functions in the case their norms are uniformly bounded (or if the functions are essentially bounded with the same bound).

The supremum of a sequence of functions is the L_1 limit of the sequence defined as $g_n = \max\{f_1, \dots f_n\}$ (the definitions of $\inf_n f_n$, $\liminf_n f_n$ and $\limsup_n f_n$ are similar). This definition is not to be confused with the pointwise definition: in $\mathsf{ACA_0}$ for a.e. x, $\sup_n f_n(x)$ exists, but it is not obvious that these pointwise values define an integrable function. Unless otherwise specified, all limits, suprema or infima, and all equalities will be meant in the L_1 sense.

Proposition 8.4 ($\mathsf{ACA_0}$) *If $\langle f_n \rangle$ is a sequence of functions in L_1 such that $|\mu(f_n)| \le M$, then* $\sup_n f_n$, $\inf_n f_n$, $\limsup f_n$ *as well as* $\liminf f_n$ *are all integrable as well.*

Proof. Consider only $\sup_n f_n$ and $\limsup_n f_n$ since the argument for $\inf_n f_n$ and $\liminf_n f_n$ is analogous. Define $g_n = \max\{f_1, \dots f_n\}$. The sequence $\langle g_n \rangle$ is increasing and $\|g_n\| \le M$. Theorem 8.1 applies: $\lim_n g_n = \sup f_n$ is an integrable function.

The argument for $\limsup_n f_n$ is similar. Define $h_{n,k} = \max_{n \le m \le k} f_n$. Then $h_{n,k} \in L_1(X)$ and $\|h_{n,k}\| \le M$ for all k, n and, for a fixed n, $\langle h_{n,k} \mid k \in \mathbb{N} \rangle$ is increasing. By the monotone convergence theorem, for all n, $\lim_k h_{n,k}$ exists and is in $L_1(X)$. Set $h_n = \lim_k h_{n,k} = \sup_{k \ge n} f_k$, then $h_n \in L_1(X)$ for all n.

Applying a similar argument to the sequence h_n (decreasing sequence of functions on $L_1(X)$, bounded in measure), it follows that $\lim_n h_n$ exists and is in $L_1(X)$. But this limit is $\limsup_n f_n$, hence $\limsup_n f_n \in L_1(X)$, as required. □

Proposition 8.4 and Lemma 7.3, imply (in ACA_0) that if a sequence $\langle f_n \rangle$ converges pointwise to an integrable function f, then $\liminf_n f_n = f$.

Corollary 8.5 (ACA_0) *(***Fatou's Lemma***) Let $\langle f_n \rangle$ be a sequence of nonnegative integrable functions with $\mu(f_n) \le M$ for all n. Then*

$$\mu(\liminf_n f_n) \le \liminf_n \mu(f_n).$$

Proof. Let $g_n = \inf_{k \ge n} f_k$. Then by Proposition 8.4, $g_n \in L_1(X)$ for each n, so $\langle g_n \rangle$ is an increasing sequence of integrable functions such that $\mu(g_n) \le \mu(f_n) \le M$ for all n. Since

$$\forall n \; (\mu(g_n) \le \mu(f_n)) \to \liminf_n \mu(g_n) \le \liminf_n \mu(f_n)$$

and $\liminf_n \mu(g_n) = \lim_n \mu(g_n) = \mu(\liminf_n f_n)$, it follows that

$$\mu(\liminf_n f_n) \le \liminf_n \mu(f_n)$$

as required. □

Now we can complete the proof of Lemma 7.4, promised in the last section.

Proof. First observe that $|f_n| + |f| - |f - f_n| \to 2|f|$ pointwise. According to the comment after Proposition 8.4, $\liminf_n(|f_n| + |f| - |f - f_n|) = 2|f|$ and by Fatou's Lemma

$$\begin{aligned} 2\mu(|f|) &= \mu(\liminf_n |f_n| + |f| - |f - f_n|) \le \liminf_n \mu(|f_n| + |f| - |f - f_n|) \\ &\le \mu(|f|) + \lim_n \mu(|f_n|) + \liminf_n(-\mu(|f - f_n|)) \\ &= 2\mu(|f|) - \limsup_n \mu(|f - f_n|). \end{aligned}$$

The last inequality yields $0 \le \limsup_n \mu(|f - f_n|) \le 0$, and it is easy to show that this implies $\lim_n \mu(|f - f_n|) = 0$. □

9 Conclusion

Although setting the foundations for work in measure theory is more difficult than in usual mathematical practice (definition of products serves as an excellent example of this), once the set-up is complete, many of the standard theorems follow, some in weak fragments of second-order arithmetic, like RCA_0 and WWKL_0. Though it is conceivable that there is a more natural approach to these topics, the current approach is capable of dealing with measure theory in a, for the most part, straightforward way.

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