Math 245
Spring 2010

1 Relations and functions practice problems

1.1 Relations

1. Determine which of the four properties: reflexive, symmetric, antisymmetric, and transitive, apply to each of the following relations on the set of integers. For each relation that is an equivalence relation, describe the equivalence classes.

   a R b iff

   (a) $a = b$
   (b) $a < b$
   (c) $a \leq b$
   (d) $a|b$ (a divides b)
   (e) $|a| = |b|$
   (f) $a^2 + a = b^2 + b$
   (g) $a < |b|$
   (h) $ab > 0$
   (i) $a + b > 0$

2. Do the same as in the previous problem for the following relations on the set of all people.

   p R q iff

   (a) p "is a father of" q
   (b) p "is a sister of" q
   (c) p "is a friend of" q
   (d) p "is an aunt of" q
   (e) p "has the same height" as q
   (f) p "likes" q
   (g) p "is married to" q

3. Let A = the set of all strings of English letters. Determine whether the following binary relations on A are reflexive, symmetric, transitive, and/or antisymmetric:

   (a) $R_1 = \{(x, y) : x \text{ and } y \text{ are strings with no letters in common}\}$
   (b) $R_2 = \{(x, y) : x \text{ and } y \text{ are strings of different lengths}\}$

(c) $R_3 = \{(x, y) : x \text{ is a longer string than } y\}$.

4. Give an example of a set A and a relation R on A which is

   (a) reflexive and symmetric, but not transitive;

   (b) reflexive and transitive, but not symmetric;

   (c) symmetric and transitive, but not reflexive. [Hint: try to draw charts to reflect the right properties]

5. Give an example of a set A and a relation R on A which is

   (a) both symmetric and antisymmetric;

   (b) neither symmetric nor antisymmetric;

   (c) can you make a conjecture describing exactly which relations on A will be both symmetric and antisymmetric in general?

6. Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?

   (a) $\{(f, g) | f(1) = g(1)\}$

   (b) $\{(f, g) | f(0) = g(0) \text{ or} f(1) = g(1)\}$

   (c) $\{(f, g) | f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$.

7. (Important example) Show that a R b iff $a \equiv b(modm)$ where m is a fixed natural number is an equivalence relation on the set of integers. Describe all the equivalence classes. For homework, do some research and write a paragraph about how modular arithmetic is used in the real world.

8. (Another important example, related to lattices and Boolean algebras) Show that the relation $\leq$ defined as $a \leq b$ iff $a \wedge b \equiv a$ is a partial order on the set of propositions.

1.2 Functions

1. Find the domain and range of these functions:

   (a) The function that assigns to each bit string the difference between the number of ones and the number of zeros;

   (b) The function that assigns to each bit string twice the number of zeros in that string;

   (c) The function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits);

   (d) The function that assigns to each positive integer the largest perfect square not exceeding this integer.

2. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one:

(a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$
(b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$
(c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

3. Which functions in the previous exercise are onto?

   Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

   (a) $f(m, n) = 2m - n$
   (b) $f(m, n) = m^2 - n^2$
   (c) $f(m, n) = m + n + 1$
   (d) $f(m, n) = |m| - |n|$
   (e) $f(m, n) = m^2 - 4$

4. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$:

   (a) $f(x) = 2x + 1$
   (b) $f(x) = x^2 + 1$
   (c) $f(x) = x^3$
   (d) $f(x) = (x^2 + 1)/(x^2 + 2)$.