Research Statement

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1 Background

My research is in two different areas: mathematical logic and mathematics education. My dissertation work was in the former, and in the past three years I have trained and conducted research in the latter. Below, I will describe my research programs in both areas, as I plan to pursue both.

2 Mathematics education

My interest in mathematics education is primarily in issues of equity, mathematical identity of students, and teaching mathematics for social justice, and related to this, curriculum development and teacher education aimed towards equity.

Between 2004-2007, I was a post-doctoral Fellow at the Center for Mathematics Education of Latinos/as (CEMELA), a multi-university consortium whose goal is improve the mathematics education of low-income Latino/a students through both outreach and research. My main role in CEMELA was co-coordinating an after-school Math Club that I helped start in the spring of 2005. I have conducted research in this setting, as part of a research team, between fall 2005 and spring 2007.

The main goal of the research is to analyze the teaching and learning of mathematics in an after-school setting, in order to identify strategies that provide effective learning opportunities to all, but particularly low-income and minority students.

The Math Club took place at an elementary school in a predominantly Latino/a border community. The activities in the Math Club, mostly projects, were usually based on the students' everyday experiences, and often had a social justice component. The setting strongly encouraged a collaborative relationship among the students.

The research is based on the premise that teaching and learning are social processes, as espoused by Vygotsky (1978), who argued that when individuals learn in a social situation, learning is enhanced and expanded. Another important idea is that the students' homes and communities are sources of funds of knowledge (Moll & Gonzalez, 2004) that students could potentially bring to learning situations if the learning environment encouraged it. Researchers in critical pedagogy (Gutstein, 2005) claim that these funds of knowledge should be given the same value as traditional "academic" knowledge.

In our papers (Díez-Palomar, Simic, & Varley, 2006; Simic-Muller, Varley & Díez-Palomar, 2007; Varley, Turner, Simic-Muller & Díez-Palomar, 2007) we have argued that the type of setting that promotes collaborative learning and uses students' funds of knowledge, positioning them as experts, has an effect on their perceptions both of mathematics and of themselves as learners. They begin to see mathematics as omnipresent, and as a tool for understanding their world. As a consequence, their participation and engagement, both in Math Club and in school is increased. They begin to seem themselves as "good at math," are able to use multiple strategies for solving problems and gladly participate in discussions about mathematics and problem solving.

2.1 Future plans

During the time I was involved in the after-school Math Club, our research team collected a wealth of data, which we are still analyzing. Currently, we are working on a book chapter about the experiences and changes in beliefs of the undergraduate facilitators, especially those who were pre-service teachers, in Math Club. The book chapter will be an extension of the recent conference presentations. I am also planning on writing an article for a practitioner journal, describing some of the projects we created.

As for future research projects, due to the experience I have had in CEMELA, I am interested in conducting research in out-of-school settings. In particular, I am interested in creating learning environments that encourage critical thinking and explore applications of mathematics, and that would primarily be geared towards women, students of color, and low-income students. I would be interested in following the changes in beliefs and performance of the students participating in these learning environments.

I am also interested in curriculum development: I would like to create curricula for various grade levels that would be based on actual applications of mathematics in sciences, economics and everyday life, and that would address social issues, such as wealth distribution or pollution. I would like to examine how this type of curriculum influences student learning, participation, and identity as mathematics learners, and also how it influences the teachers' beliefs and teaching styles.

Finally, in the future, I would also to learn more about participation of women in mathematics and other sciences and conduct research in this area.

2.2 References

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Gutstein, E. (2005). *Reading and writing the world with mathematics: Toward a pedagogy for social justice.* New York: Routledge.

Moll, L. & Gonzalez, N (2004). Engaging life: A funds of knowledge approach to multicultural education. In J. Banks & C. McGee Banks (Eds.). *Handbook of research on multicultural education*, (2nd Ed.) (pp. 669-715). NY: Jossey-Bass.

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Vygotsky, L.S. (1978). *Mind in society.* Cambridge, MA: Harvard University Press

3 Mathematical logic

My main area of interest in mathematical logic are the foundations of mathematics, more precisely, formalizing portions of mathematics in subsystems of second-order arithmetic.

Full second-order arithmetic, often denoted as Z_2, consists of basic axioms for the natural numbers as an ordered semi-ring, the induction axiom and the comprehension scheme

$$\exists X\ \forall n\ [n \in X \leftrightarrow \varphi(n)]$$

where φ is any formula of the language in which X doesn't occur freely. In most cases, full comprehension is more than is needed, and can be replaced with weaker set existence axioms, yielding different subsystems of second-order arithmetic. Induction is also restricted. Most countable mathematics can be formalized in one of the systems obtained this way. Moreover, for the most part, six subsystems suffice. Below, I list only those relevant to my work. They are ordered from weakest to strongest. The subscript 0 stands for restricted induction.

- [RCA_0]: Basic axioms, plus Σ^0_1 induction and Δ^0_1 comprehension.
- [WWKL_0]: RCA_0 plus weak-weak König's Lemma ($WWKL$), stating that if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then

$$\lim_n \frac{|\{\sigma \in T | length(\sigma) = n\}|}{2^n} = 0.$$

- [ACA_0]: RCA_0 plus comprehension for all arithmetic formulas (ACA).

The area of research I pursue is frequently called *reverse mathematics*, since we not only prove that the axioms imply certain theorems, but that the theorems are logically equivalent to set existence axioms used to prove them. For example, the Bolzano-Weierstrass theorem is equivalent to (ACA) over RCA_0.

The first challenge one faces is in choosing the suitable formalization of familiar concepts in the restricted language of second-order arithmetic. We must always keep in mind that all objects considered must be represented either as countable sequences of natural numbers or completions of such sequences.

For example, a real number is represented as a strong Cauchy sequence of rational numbers, that is, $\langle q_k \mid k \in \mathbb{N}\rangle$ such that $\forall k\ \forall j\ |q_k - q_{k+j}| \leq 2^{-k}$.

We always try to stay as faithful to standard formalization as possible, but where this is difficult, finding the most suitable alternative can prove just as interesting.

The most comprehensive resource on reverse mathematics is Simpson's monograph [4].

3.1 Some definitions and statements of theorems

Definition 3.1 *The code for a real separable Hilbert space consists of a countable vector space A over $\mathbb{Q}$ together with a function $\langle\cdot,\cdot\rangle : A \times A \to \mathbb{R}$ satisfying*

$$\langle x, x\rangle \geq 0 \quad \langle x, y\rangle = \langle y, x\rangle \quad \langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$$

for all $x, y, z \in A$ and $a, b \in \mathbb{Q}$. The points of H are strong Cauchy sequences of points in A.

The idea is to start with a countable set equipped with an inner product and then take the completion. Complete separable metric spaces and Banach spaces are defined analogously.

Definition 3.2 *An* isometry *on a Banach (or Hilbert) space $H = \bar{A}$ is defined as a sequence $T : A \to H$ that satisfies $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in \mathbb{Q}$, $x, y \in A$ and $\|Tx\| = \|x\|$ for all $x \in A$. For an arbitrary $x = \langle x_n \rangle$, $T(x)$ is defined as $\lim_n T(x_n)$.*

Theorem 3.3 (Mean Ergodic Theorem) *Let H be a Hilbert space and $T : H \to H$ an isometry. The sequence $\langle S_n x \rangle$, where $S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$ converges for all $x \in H$.*

Theorem 3.4 (Pointwise Ergodic Theorem) *Let $T : L_1(X) \to L_1(X)$ be an isometry, where X is a compact metric space. The sequence $\langle S_n f(x) \rangle$, where S_n is defined as above, converges a.e. for all $f \in L_1(X)$.*

3.2 Results

The main goal of my dissertation work, supervised by Jeremy Avigad, Associate Professor at Carnegie Mellon University, was to formalize and prove the mean and pointwise ergodic theorems in the framework of second-order arithmetic. During this process, many new problems presented themselves. For example, in order to prove the mean ergodic theorem, it is necessary to obtain various results for Hilbert spaces. The results regarding Hilbert spaces and the mean ergodic theorem can be found in [1]. Similarly, the proof of the pointwise ergodic theorem raises the question of what the best and most natural representation of L_p spaces is. These results can be found in [3]. The entire dissertation [2] is available at http://math.arizona.edu/∼ksimic.

Some of the results I established are:

- [RCA_0] Every Hilbert space has an orthonormal basis.
- Existence of orthogonal projection on a closed subspace of a Hilbert space is equivalent to (ACA).
- [RCA_0] The kernel of a bounded linear functional is a closed subspace.
- The Riesz representation theorem for Hilbert spaces is equivalent to (ACA).
- The mean ergodic theorem is equivalent to (ACA).
- [RCA_0] Convergence of $\langle S_n \rangle$ in L_2 norm implies its convergence in L_1 norm.
- [ACA_0] Fatou's lemma and the Monotone convergence theorem hold (moreover, the latter is equivalent to (ACA)). This result was cited in [5] be Terence Tao.
- The pointwise ergodic theorem for a multiplicative isometry T that preserves the supremum norm is equivalent to (ACA).

3.3 Research Plan

In the future, I plan to unite two of my main interest in mathematical logic: reverse mathematics and constructive mathematics (mathematics in which the underlying logic is intuitionist), and to explore the connections between the two. Which results of reverse mathematics established to date can be formalized in intuitionistic logic? Which modifications need to be made in the proofs in doing so? I am particularly interested in results pertaining to functional analysis, measure theory, and, most recently, nonstandard analysis. Not much literature currently exists on the subject of nonstandard analysis from the constructivist perspective and in subsystems of second-order arithmetic.

I also intend to continue work in measure theory, and would like to look into reverse mathematics of probability theory.

References

[1] Jeremy Avigad and Ksenija Simic. Fundamental notions of analysis in subsystems of second-order arithmetic. *Annals of Pure and Applied Logic*, 139:138–184, 2006.

[2] Ksenija Simic. *Aspects of ergodic theory in subsystems of second-order arithmetic*. PhD thesis, Carnegie Mellon University, 2004.

[3] Ksenija Simic. The pointwise ergodic theorem in subsystems of second-order arithmetic. *Journal of Symbolic Logic*, 72:46–66, 2007.

[4] Stephen G. Simpson. *Subsystems of Second-Order Arithmetic*. Springer, Berlin, 1998.

[5] Terence Tao. Norm convergence of multiple ergodic averages for commuting transformations. Available at http://www.arxiv.org/.