Three Proofs of the Cauchy-Buniakowski-Schwarz Inequality

Theorem 1 (The Cauchy-Buniakowski-Schwarz Theorem) *If* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Equality holds exactly when one vector is a scalar multiple of the other.

Proof I. If either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u}\| \, \|\mathbf{v}\| = 0$ so equality holds. For the remainder of the proof, we will assume that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

Let α and β be arbitrary scalars. Then $\|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 \geq 0$.

Using properties of lengths and dot products,

$$\begin{aligned} \|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 &= (\alpha\mathbf{u} + \beta\mathbf{v}) \cdot (\alpha\mathbf{u} + \beta\mathbf{v}) \\ &= \alpha^2 \mathbf{u} \cdot \mathbf{u} + \alpha\beta \mathbf{u} \cdot \mathbf{v} + \beta\alpha \mathbf{v} \cdot \mathbf{u} + \beta^2 \mathbf{v} \cdot \mathbf{v} \\ &= \alpha^2 \|\mathbf{u}\|^2 + 2\alpha\beta \mathbf{u} \cdot \mathbf{v} + \boldsymbol{\beta}^2 \|\mathbf{v}\|^2 \end{aligned}$$

Since this holds for all scalars α and β, we are free to choose $\alpha = \|\mathbf{v}\|$ and $\beta = \mp \|\mathbf{u}\|$. Substituting,

$$\begin{aligned} \|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 &= \|\mathbf{v}\|^2 \|\mathbf{u}\|^2 + 2 \|\mathbf{v}\| (\mp \|\mathbf{u}\|) \, \mathbf{u} \cdot \mathbf{v} + (\mp \|\mathbf{u}\|)^2 \|\mathbf{v}\|^2 \\ &= 2 \|\mathbf{v}\| \, \|\mathbf{u}\| \, (\|\mathbf{v}\| \, \|\mathbf{u}\| \mp \mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, $\|\mathbf{u}\| > 0$ and $\|\mathbf{v}\| > 0$, so $\|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 \geq 0$ is true exactly when $\|\mathbf{v}\| \, \|\mathbf{u}\| \mp \mathbf{u} \cdot \mathbf{v} \geq \mathbf{0}$. That is, exactly when $\|\mathbf{v}\| \, \|\mathbf{u}\| \geq \pm\mathbf{u} \cdot \mathbf{v}$, which is the same as $\|\mathbf{v}\| \, \|\mathbf{u}\| \geq |\mathbf{u} \cdot \mathbf{v}|$.

Note that $\|\mathbf{v}\| \, \|\mathbf{u}\| = |\mathbf{u} \cdot \mathbf{v}|$ exactly when $\|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 = 0$, which is exactly when $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$. Since $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ implies either $\alpha = \beta = 0$ or else $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other. Since α and β are nonzero, one vector must be a scalar multiple of the other.

Proof II. If either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u}\| \|\mathbf{v}\| = 0$ so equality holds. For the remainder of the proof, we will assume that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

First, suppose that $\mathbf{x}$ and $\mathbf{y}$ are unit vectors. Using properties of lengths and dot products,

$$\begin{aligned} \|\mathbf{x} \pm \mathbf{y}\|^2 &= (\mathbf{x} \pm \mathbf{y}) \cdot (\mathbf{x} \pm \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} \pm \mathbf{2x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|^2 \pm 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\ &= 1 \pm 2\mathbf{x} \cdot \mathbf{y} + 1 \\ &= 2\,(1 \pm \mathbf{x} \cdot \mathbf{y}) \end{aligned}$$

Since $\|\mathbf{x} \pm \mathbf{y}\|^2 \geq 0$, $1 \pm \mathbf{x} \cdot \mathbf{y} \geq 0$. This is the same as $1 \geq \mp \mathbf{x} \cdot \mathbf{y}$. Thus, $|\mathbf{x} \cdot \mathbf{y}| \leq 1$. Further, equality holds exactly when $\mathbf{x} \pm \mathbf{y} = \mathbf{0}$, which means that $\mathbf{y} = \pm \mathbf{x}$.

Now suppose that $\mathbf{u}$ and $\mathbf{v}$ are general nonzero vectors. Then $\|\mathbf{u}\| > 0$ and $\|\mathbf{v}\| > 0$ so $\mathbf{x} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ and $\mathbf{y} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ are unit vectors. Then

$$\begin{aligned} |\mathbf{x} \cdot \mathbf{y}| &\leq 1 \\ \left| \left(\frac{1}{\|\mathbf{u}\|}\mathbf{u} \right) \cdot \left(\frac{1}{\|\mathbf{v}\|}\mathbf{v} \right) \right| &\leq 1 \\ \frac{1}{\|\mathbf{u}\|} \frac{1}{\|\mathbf{v}\|} |\mathbf{u} \cdot \mathbf{v}| &\leq 1 \\ |\mathbf{u} \cdot \mathbf{v}| &\leq \|\mathbf{u}\| \|\mathbf{v}\| \end{aligned}$$

Equality holds exactly when $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \pm \frac{1}{\|\mathbf{v}\|}\mathbf{v}$, which means exactly when one vector is a scalar multiple of the other.

Proof III. Recall that when a, b, c are real numbers with $a \neq 0$, the quadratic function at^2+bt+c has at most one real root exactly when $b^2-4ac \leq 0$. Further, in this case, $at^2+bt+c=0$ for some real t occurs exactly when $b^2-4ac=0$.

If either $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v}=0$ and $\|\mathbf{u}\|\,\|\mathbf{v}\|=0$, so the desired equality holds. For the remainder of the proof, we will assume that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

Consider the function $f(t)=\|t\mathbf{u}+\mathbf{v}\|^2$. Clearly, $f(t)\geq 0$ for all $t\in\mathbb{R}$. Further, $f(t)=0$ for some real t exactly when $\|t\mathbf{u}+\mathbf{v}\|^2=0$, which is to say, when $\mathbf{v}=-t\mathbf{u}$ for some real number t. That is, equality holds exactly when one vector is a scalar multiple of the other.

For all real t,

$$\begin{aligned}\|t\mathbf{u}+\mathbf{v}\|^2 &= (t\mathbf{u}+\mathbf{v})\cdot(t\mathbf{u}+\mathbf{v}) = t^2\mathbf{u}\cdot\mathbf{u}+2t\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 t^2+(2\mathbf{u}\cdot\mathbf{v})t+\|\mathbf{v}\|^2\end{aligned}$$

Let $a=\|\mathbf{u}\|^2>0$ since $\mathbf{u}$ is nonzero; let $b=2\mathbf{u}\cdot\mathbf{v}$; and let $c=\|\mathbf{v}\|^2$. The condition $b^2-4ac\leq 0$ is equivalent to $(2\mathbf{u}\cdot\mathbf{v})^2-4\|\mathbf{u}\|^2\|\mathbf{v}\|^2\leq 0$. Equivalently, $(\mathbf{u}\cdot\mathbf{v})^2\leq\|\mathbf{u}\|^2\|\mathbf{v}\|^2$. Taking the square root of each side produces the desired inequality. Equality occurs in each of these inequalities exactly when $\|t\mathbf{u}+\mathbf{v}\|^2=0$ for some real t, which is to say, when one vector is a scalar multiple of the other.