Five Fundamental Results from Math 151

Note that for real numbers a and b with $a < b$, (a, b) denotes the set of all real numbers x satisfying $a < x < b$, and $[a, b]$ denotes the set of all real numbers x satisfying $a \leq x \leq b$. That is, (a, b) is a segment of the number line without its endpoints, and $[a, b]$ is a segment of the number line including its endpoints. A function $f(x)$ is said to be continuous on $[a, b]$ if it is continuous at each x in $[a, b]$. A function $f(x)$ is said to be differentiable on (a, b) if the derivative $f'(x)$ is defined at each x in (a, b).

The first two results only require continuity.

Theorem 1 ***(The Intermediate Value Theorem - IVT)*** *Let $a < b$ be real numbers. Let f be a function that is continuous on $[a, b]$. Then f passes through every value between $f(a)$ and $f(b)$ at least once on $[a, b]$.. (That is, f attains every value intermediate to $f(a)$ and $f(b)$.) Equivalently, for every real number y between $f(a)$ and $f(b)$, there is at least one c between a and b such that $y = f(c)$.*

Theorem 2 ***(Max-Min Theorem)*** *Let $a < b$ be real numbers. Let f be a function that is continuous on $[a, b]$. Then f attains a maximum and a minimum on $[a, b]$. That is, there exists at least one point $x_{\min}$ and at least one point $x_{\max}$ in $[a, b]$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all x in $[a, b]$ where both $f(x_{\min})$ and $f(x_{\max})$ are real numbers (ie. NOT plus or minus infinity).*

The next result requires both continuity and differentiability.

Theorem 3 ***(The Mean Value Theorem for Derivatives - MVT4D)*** *Let $a < b$ be real numbers. Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one c between a and b at which*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \ .$$

That is, there is at least one c between a and b at which the tangent line to $f(x)$ is parallel to the line through the points $(a, f(a))$ and $(b, f(b))$.

The next two results are consequences of the MVT4D:

Corollary 4 *Let $a < b$ be real numbers. Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all x in (a, b), then $f(x)$ is constant on the interval $[a, b]$.*

Corollary 5 *Let $a < b$ be real numbers. Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ for all x in (a, b), then there is a constant k such that $f(x) = g(x) + k$ for all x in $[a, b]$.*