The Complex Numbers and Their Arithmetic

Previously, we have discussed $\mathbb{R}$, the set of all real numbers, and the fundamental properties satisfied by real number arithmetic. We have also discussed how the problem of solving $x^2+1=0$, or equivalently, factoring x^2+1, led to the creation (discovery?) of the number i that satisfies $i^2=(-i)^2=-1$. We saw that the general problem of solving quadratic equations with real coefficients, or equivalently, factoring quadratic polynomials with real coefficients, showed the need for set of numbers that contained more than just the real numbers. We investigate that set here.

The set of all numbers of the form $a+bi$ where a and b are real numbers is called the set of complex numbers, and it is denoted by $\mathbb{C}$. To use the formal notation for sets,

$$\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}\,.$$

Each of the following is a complex number:

$$3+2i$$

$$\sqrt{\pi}-\frac{e}{7.9}i$$

$$\begin{aligned} 1 &= 1+0i \\ 0 &= 0+0i \\ i &= 0+1i \end{aligned}$$

Just as we thought of subtraction of a real number r as addition of $-r$, we often find it convenient to switch between the two equivalent expressions $a-bi$ and $a+(-b)i$ where a and b are real numbers. Thus, we might sometimes write $3-2i$, and other times write $3+(-2)i$.

Notice that our old friend $\mathbb{R}$, the set of all real numbers is actually a subset of $\mathbb{C}$ since $a\in\mathbb{R}$ implies $a=a+0i$, which is in $\mathbb{C}$. Using set notation, $\mathbb{R}\subset\mathbb{C}$. We sometimes speak of $\mathbb{C}$ as consisting of the real numbers together with the *nonreal* complex numbers. That is,

$$\mathbb{C}=\mathbb{R}\cup\{a+bi : a,b\in\mathbb{R}\ \textit{and}\ b\neq 0\}\,.$$

A special subset of $\mathbb{C}$ is the set of *imaginary numbers*, which is the set of all numbers of the form $0+bi$ where b is real. Thus $i, \log(\sqrt{\pi})i$, $-\frac{8}{13.5}i$ and $5i$ are all imaginary numbers.

It is common to use the letters w and z to stand for complex numbers. For the complex number $z=\ a+bi$, the *real part* of z is the real number a, and the *imaginary part* of z is the real number b. These parts are often abbreviated $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. Thus, $z=\mathrm{Re}(z)+\mathrm{Im}(z)\,i$. Notice that the imaginary part is NOT imaginary, it is real.

How do we do arithmetic? The short answer is the same as always, except that whenever i^2 occurs, we can replace it with -1. For convenience, in the following discusion we will use the complex numbers $z = a + bi$ and $w = c + di$ where a, b, c, d are all real. Then

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Since we are actually adding real numbers in the parentheses, we easily obtain associativity and commutativity for complex addition and easily show that the set of all complex numbers is closed under addition.

$$z + 0 = (a + bi) + (0 + 0i) = (a + 0) + (b + 0)i = a + bi = z$$

so 0 is the additive identity element for the set of all complex numbers. It is easy to check that $-z = (-a) + (-b)i$ satisfies $z + (-z) = 0$, so each complex number has an additive inverse, and the additive inverse is itself a complex number. As we did with real numbers, we formally define subtraction of complex numbers in terms of addition of the additive inverse. Thus,

$$z - w = z + (-w) = (a + bi) + ((-c) + (-d)i) = (a - c) + (b - d)i.$$

Exercise 1 *Suppose that $z = 3 - 8i$ and that $w = -5 + 2i$.*

1. *Find* $\mathrm{Re}(w)$ *and* $\mathrm{Im}(w)$.
2. *Compute $z + z$, $z + w$, $z - w$, $w - z$.*
3. *What is the additive inverse of w?*
4. *Solve $a + bi + 4 + 2i = 9 - 3i$ where a and b are real*
5. *Let $v = a + bi$ where a and b are real. Simplify $(a + bi) + (a - bi)$ and $(a + bi) - (a - bi)$, and express your results in terms of* $\mathrm{Re}(v)$ *and* $\mathrm{Im}(v)$ *as needed.*

We turn next to multiplication. Remember how you multiplied $(a+bx)(c+dx)$ to get $ac+bcx+adx+bdx^2$? We will use that approach to multiply complex numbers.

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= ac + (bi)c + a(di) + (bi)(di) \\ &= ac + (bc)i + (ad)i + (bd)i^2 \\ &= ac + (bc + ad)i + bd(-1) \\ &= (ac - bd) + (bc + ad)i \end{aligned}$$

Notice that $ac - bd$ and $bc + ad$ are real numbers! Since we are actually working with real numbers, it is fairly straight forward to obtain associativity and commutativity for complex number multiplication and to show that the set of all complex numbers is closed under complex number multiplication.

Exercise 2 *Suppose that a and b are real numbers.*

1. *Compute* zw, wz,*and* $z^2 = zz$ *when* $z = 3 - 8i$ *and* $w = -5 + 2i$.
2. *Let* r *be a real number. By using* $r = r + 0i$, *confirm that* $r(a + bi) = (ra) + (rb)i$.
3. *Using* $2 = 2 + 0i$, *determine if it is true that* $z + z = 2z$ *where* z *is the complex number in the first problem of this set.*
4. *Show that* $(a + bi)^2$ *is real exactly when at least one of* $a = 0$ *and* $b = 0$ *holds.*
5. *Compute* $(4 + 4i)^2$ *and* $4^2 + (4i)^2$.
6. *Show that* $(a + bi)(a - bi)$ *is always real.*
7. *Show that* $(a + bi)(a - bi)$ *is always a nonnegative real number, and that it is positive unless* $a = b = 0$.

From the previous set of exercises, we know that $1z = z$ for all complex numbers z, so 1 is the multiplicative identity element for the set of all complex numbers.

Suppose that at least one of the real numbers a and b is nonzero. Then $a^2 + b^2$ is positive. Consider the product:

$$\begin{aligned}
(a+bi)\left(\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i\right) &= \left(a\,\frac{a}{a^2+b^2}-b\,\frac{-b}{a^2+b^2}\right)+\left(a\,\frac{-b}{a^2+b^2}+b\,\frac{a}{a^2+b^2}\right)i \\
&= \frac{a^2+(-b)^2}{a^2+b^2}+\frac{-ab+ba}{a^2+b^2}i \\
& \quad \frac{a^2+b^2}{a^2+b^2}+\frac{-ab+ab}{a^2+b^2}i \\
&= 1+0i
\end{aligned}$$

Apparently, when $z = a + bi \neq 0$, z has a multiplicative inverse z^{-1} given by

$$z^{-1} = (a + bi)^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$

For example,

$$3^{-1} = (3 + 0i)^{-1} = \frac{3}{3^2 + 0^2} + \frac{-0}{3^2 + 0^2}i = \frac{3}{9} + 0i = \frac{1}{3}$$

As we did for real numbers, we will define division by a nonzero complex number as multiplication by its multiplicative inverse. That is, if w is a nonzero complex number, then

$$\frac{z}{w} = z(w^{-1}).$$

As an example,

$$\frac{1}{i} = i^{-1} = (0+1i)^{-1} = \frac{0}{0^2+1^2} + \frac{-1}{0^2+1^2}i = 0 + (-1)i = -i$$

This last result should not be surprising since it tells us that

$$\frac{i}{i} = i(i^{-1}) = i(-i) = -i^2 = -(-1) = 1,$$

as we would have hoped.

Exercise 3 *Find expressions in standard form for* $(3+2i)^{-1}$ *and for* $(8+7i)/(3+2i)$.

Exercise 4 *Show that if* a, b, c, d, g, h *are real numbers, and if* $z = a + bi$, $w = c + di$ *and* $v = g + hi$, *then*

$$v(w+z) = vw + vz.$$

The preceding exercise confirms that complex number addition and multiplication distribute. Thus, as a set, the complex numbers satisfy the same 11 fundamental properties as the set of real numbers do!

Let us re-examine the expression for z^{-1}. Using a little arithmetic, we have

$$z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i = \frac{1}{a^2+b^2}\left(a - bi\right).$$

It turns out that for a complex number $a+bi$, the two numbers a^2+b^2 and $a-bi$ are very important. As we remarked before, a^2+b^2 is always positive except when $a = b = 0$. We define the *magnitude* of the complex number $z = a + bi$ to be the nonnegative real number $\sqrt{a^2+b^2}$. Every nonzero complex number has positive magnitude. We denote the magnitude by $|z|$. That notation might look familiar! Suppose that r is a real number. Then its magnitude,

$$|r| = |r + 0i| = \sqrt{r^2+0^2} = \sqrt{r^2}$$

But recall that for a real number r, $\sqrt{r^2}$ is precisely the absolute value of r. Thus the magnitude of a complex number is an extension of the concept of absolute value of a real number. The following exercise shows that the magnitude has some similar properties to those of the absolute value.

Exercise 5 *Let* $z = a + bi$ *and let* $w = c + di$ *where* a, b, c, d *are real.*

1. *Show that* $|-z| = |z|$.
2. *Show that* $|zw| = |z|\,|w|$.
3. *If* c *is a positive real number, show that* $|cz| = c\,|z|$.

Exercise 6 *Compute* $|1|, |-1|, |i|, |-i|, |3+4i|, |3-4i|, |-3+4i|$ *and* $|-4-3i|$.

For a complex number $z = a + bi$, the *conjugate* of z is the complex number $\overline{z} = a - bi$. The notation $\overline{z}$ is read aloud as either "z bar" or "z conjugate". Some examples are: $\overline{0} = 0$, $\overline{-3+7i} = -3-7i$, and $\overline{i} = -i$. Notice that for a real number $r = r + 0i$, $\overline{r} = r - 0i = r$. Also notice that

$$\overline{(\overline{z})} = \overline{a - bi} = a - (-b)i = a + bi = z.$$

On crucial property of complex conjugation is that it distributes with addition and multiplication. Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}
\overline{z} + \overline{w} &= \overline{a+bi} + \overline{c+di} \\
&= (a-bi) + (c-di) \\
&= (a+c) + ((-b)+(-d))\,i \\
&= (a+c) - (b+d)i \\
&= \overline{(a+c)+(b+d)i} = \\
&\quad \overline{z+w}
\end{aligned}$$

$$\begin{aligned}
\overline{z}\,\overline{w} &= \left(\overline{a+bi}\right)\left(\overline{c+di}\right) \\
&= (a-bi)(c-di) \\
&= (ac-bd) - (ad+bc)i \\
&= \overline{(ac-bd)+(ad+bc)i} \\
&= \overline{(a+c)(b+d)i} \\
&= \overline{z\,w}
\end{aligned}$$

Exercise 7 *Let* $z = a + bi$.

1. *Show that* $|\overline{z}| = |z|$.
2. *Show that* $z + \overline{z} = 2\operatorname{Re}(z)$ *and that* $z - \overline{z} = 2\operatorname{Im}(z)\,i$.
3. *Show that* $z\overline{z} = |z|^2$.
4. *Show that* $\overline{z} = z$ *is equivalent to* z *is a real number.*
5. *Show that* $\overline{z} = -z$ *is equivalent to* z *is an imaginary number.*

Notice that the quadratic formula from precalculus algebra implicitly uses complex conjugates when the roots are nonreal. The expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

becomes, when $b^2 - 4ac < 0$,

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} \\ &= \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \\ &= \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}\, i\end{aligned}$$

so that the roots are

$$\begin{aligned} root\#1 &= \frac{-b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}\, i \\ root\#2 &= \frac{-b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}\, i \end{aligned}$$

with $\overline{root\#1} = root\#2$ and $\overline{root\#2} = root\#1$.

Let us summarize: The set $\mathbb{C}$ is closed under addition, subtraction, multiplication, division and complex conjugation. A complex number is nonzero if and only if its magnitude is positive. A complex number and its conjugate always have the same magnitude. The conjugate of a complex number is the number itself exactly when the number is real, otherwise the two numbers have different signs in their imaginary part.

In fact, complex numbers have wonderfully rich properties. For example, the set of complex numbers (like the set of real numbers) is closed under taking limits. That is, if a sequence of complex numbers has a limit, then the limit must be a complex number. As they say in those annoying commercials on TV, "BUT WAIT, THERE'S MORE!!!" There is the FTA – the Fundamental Theorem of Algebra. (Now would be a good time to read the handout on the Fundamental Theorem of Algebra that is located on my course webpage. Try www.plu.edu/~stuartjl/, and then follow the links to my courses page.)