Determinants – An Introduction*

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The **determinant** is a useful function that takes a square matrix as its input and returns a scalar as its output. There are many ways to define a determinant, and each of these equivalent definitions has its own advantages. The one that we will use has two advantages, first we will not need to introduce any complicated rules about permutations or about recursions, and second, our approach is close to the technique used by numerical software to compute the determinant. Interestingly, the history of the determinant is much longer than that of the matrix. Whereas matrices were first introduced in 1850, determinants were first computed perhaps as early as two thousand years ago. Books on the theory of determinants appeared in both Europe and Asia by the seventeenth century. It was only in the early twentieth century that matrices advanced in importance and that determinants receded to an auxiliary role.

1 Finding the Determinant via Row Operations

We begin our treatment of the determinant by giving a rule for the determinant of a very simple type of matrix, the triangular matrix. **If A is either an upper triangular matrix or a lower triangular matrix, then the determinant of A, denoted $\det(A)$, is exactly the product of the diagonal entries of A.** Thus, if

$$U = \begin{bmatrix} 3 & 9 & -1 \\ 0 & \sqrt{2} & \pi \\ 0 & 0 & -e^2 \end{bmatrix} \text{ and if } L = \begin{bmatrix} 1/3 & 0 & 0 \\ 9 & -2 & 0 \\ 1/2 & 0 & -4 \end{bmatrix},$$

then $\det(U) = 3(\sqrt{2})(-e^2) = -3e^2\sqrt{2}$ and $\det(L) = (1/3)(-2)(-4) = 8/3$. Also recall that every diagonal matrix is both upper triangular and lower triangular. Thus $\det(I_3) = (1)^3 = 1$.

What happens if the $n \times n$ matrix A is not a triangular matrix? It turns out that the same tools we used for Gaussian elimination - row operations and elementary matrices - will rescue us. Recall that there are three types of row operations, and that performing each operation on A has the same effect as multiplying A on the left by the corresponding elementary matrix.

Suppose that we multiply one row of A by the nonzero scalar c. This is the same as forming the product $E_I A$ where E_I is the type I elementary matrix obtained by multiplying the corresponding row of I_n by the same nonzero scalar c. How does this affect the determinant?

$$\det(E_I A) = c \det(A).$$

That is, the effect of multiplying one row by the scalar c is that the determinant is multiplied by c.

For example, multiplying the first row of U by 7 produces a new matrix

$$V = \begin{bmatrix} 21 & 63 & -7 \\ 0 & \sqrt{2} & \pi \\ 0 & 0 & -e^2 \end{bmatrix},$$

and clearly, $\det(V) = 21(\sqrt{2})(-e^2) = 7\det(U)$.

Suppose that we switch two rows of A. This is the same as forming the product $E_{II}A$ where E_{II} is the type II elementary matrix obtained by switching the corresponding rows of I_n. How does this affect the determinant?

$$\det(E_{II}A) = -\det(A).$$

That is, the effect of switching two rows is that the determinant is multiplied by -1.

For example, if A is the matrix

$$A = \begin{bmatrix} 9 & -2 & 0 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & -4 \end{bmatrix},$$

then switching the first and second rows of A produces the matrix L, and hence, $\det(L) = -\det(A)$, so $\det(A) = -8/3$.

Finally, suppose that we add a multiple of one row of A to another row of A. This is the same as forming the product $E_{III}A$ where E_{III} is the type III elementary matrix obtained by adding the same multiple of the corresponding row of I_n to another row of I_n. How does this affect the determinant?

$$\det(E_{III}A) = \det(A).$$

That is, this type of row operation has no effect on the determinant.

For example, multiplying the second row of L by 1/2 and adding it to the third row of L produces a new matrix

$$K = \begin{bmatrix} 1/3 & 0 & 0 \\ 9 & -2 & 0 \\ 5 & -1 & -4 \end{bmatrix},$$

and clearly, $\det(K) = (1/3)(-2)(-4) = 8/3 = \det(L)$.

We are now ready to compute the determinant of an arbitrary $n \times n$ matrix A. Recall that the process of Gaussian elimination uses row operations (or equivalently, multiplication on the left by elementary matrices) to transform our starting matrix into a matrix in echelon form. Also note that a square matrix in echelon form is an upper triangular matrix! That is, for any $n \times n$ matrix A, we kind find a sequence of elementary matrices $E_1, E_2, \ldots, E_{k-2}, E_{k-1}, E_k$ so that $E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A = R$ where R is an echelon form. Then

$$\det(E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A) = \det(R).$$

Noting that

$$\begin{aligned} E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A &= E_k \left(E_{k-1} E_{k-2} \cdots E_2 E_1 A\right) \\ &= E_k \left(E_{k-1} \left(E_{k-2} \cdots E_2 E_1 A\right)\right) \\ &= \cdots \\ &= E_k \left(E_{k-1} \left(E_{k-2} \left(\cdots E_2 \left(E_1 A\right)\right)\right)\right), \end{aligned}$$

it follows that

$$\begin{aligned} \det(E_k E_{k-1} \cdots E_2 E_1 A) &= \det\left(E_k \left(E_{k-1} E_{k-2} \cdots E_2 E_1 A\right)\right) \\ &= \det\left(E_k \left(E_{k-1} \left(E_{k-2} \cdots E_2 E_1 A\right)\right)\right) \\ &= \cdots \\ &= \det\left(E_k \left(E_{k-1} \left(E_{k-2} \left(\cdots \left(E_2 \left(E_1 A\right)\right)\right)\right)\right)\right) \\ &= \prod_{E_j \ type\ I} (scalars) \bullet (-1)^{\#\ of\ switches} \bullet \det(A) \end{aligned}$$

where $\prod(scalars)$ denotes the product of all of the *nonzero* scalars that appear in elementary matrices of type E_I, and where $\#\ of\ switches$ denotes the number of elementary matrices of type E_{II}. (Note: Even though the sequence of row operations used to transform a matrix A to R is not unique, it turns out that $\prod(scalars)$ and $(-1)^{\#\ of\ switches}$ depend only on A and R, and not on the particular sequence of row operations.) Thus, we have

$$\det(R) = \prod(scalars) \bullet (-1)^{\#\ of\ switches} \bullet \det(A).$$

Summarizing,

Theorem 1 *Let A be an $n \times n$ matrix. Let R be an echelon form for A. then*

$$\det(A) = \left[(-1)^{\#\ of\ switches}\right] \frac{\prod_{j=1}^{n} r_{jj}}{\prod(scalars)}$$

for any sequence of elementary matrices that transforms A into R.

Notice why it's important the scalars used in the type I elementary row operations are nonzero?

Example 2 *If A is the 1×1 matrix $A = [a]$, then clearly,* $\det(A) = a$.

Example 3 *Let A be the matrix*

$$A = \begin{bmatrix} 0 & 1/3 & 1/5 & 2/3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 8 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

Perform the following sequence of operations to transform A. Swap rows 1 and 2.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1/3 & 1/5 & 2/3 \\ 1 & 4 & 4 & 8 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

Subtract row 1 from row 3. Subtract twice row 1 from row 4.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1/3 & 1/5 & 2/3 \\ 0 & 2 & 1 & 4 \\ 0 & -2 & -4 & -6 \end{bmatrix}$$

Multiply row 2 by 3. Add row 3 to row 4.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3/5 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Subtract two copies of row 2 from row 3.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3/5 & 2 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Finally, subtract 15 copies of row 3 from row 4 to obtain an echelon form R:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3/5 & 2 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Notice that $\det(R) = (1)(1)(-1/5)(-2) = 2/5$, that we used one row swap, and that we multiplied a row by a scalar once, using the scalar 3. Thus

$$\det(A) = (-1)^1 \frac{2/5}{3} = -\frac{2}{15}.$$

Exercise 4 *Compute the determinants of the following matrices:*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 11 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 11 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Exercise 5 *Do the unspecified values denoted by* $*, @$ *or* $\%$ *affect the value of the determinant of*

$$M = \begin{bmatrix} 0 & 0 & 11 \\ 0 & -2 & * \\ 3 & @ & \% \end{bmatrix}?$$

Explain your answer.

Exercise 6 *Suppose that* A *is a* 3×3 *matrix with* $\det(A) = 7$. *Find the value of* $\det(5A)$. *(Hint: What do we have to do to the rows of* A *to obtain* $5A$ *?)*

Exercise 7 *Suppose that* A *is an* $n \times n$ *matrix and that* k *is a scalar. Express the value of* $\det(kA)$ *in terms of* $\det(A)$ *and* k. *Justify your answer.*

Exercise 8 *By carefully considering two cases:* $a \neq 0$ *and* $a = 0$, *show that*

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

for all scalars a, b, c *and* d.

Exercise 9 *Show that for all scalars* a, b, c

$$\det\left(\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}\right) = b - a$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (c-a)(c-b)(b-a)$$

What restrictions on a, b, c *guarantee that these determinants are nonzero?*

2 Important Properties of the Determinant

2.1 Testing for Inverses via the Determinant

At the start of this set of notes, we said that determinants are useful. The following result gives one important use of the determinant, testing whether a matrix has an inverse.

Theorem 10 *Let A be an $n \times n$ matrix. Then A is invertible if and only if* $\det(A) \neq 0$.

Proof. Let R be the reduced row echelon form of A. Recall that A is invertible if and only if its reduced row echelon form is the identity matrix. Clearly, if R is the identity matrix, then $\det(R) = (1)^n = 1$. If R is not the identity matrix, then it is upper triangular with at least one zero entry on the diagonal, and hence, $\det(R) = 0$. Since the terms $\prod(scalars)$ and $(-1)^{\#\ of\ switches}$ in the determinant formula given by Theorem 1 are always nonzero, $\det(A)$ is nonzero exactly when $\det(R)$ is nonzero. ■

The matrices U, L, V, K and A in the previous section are all invertible since each has a nonzero determinant. On the other hand, the matrix $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$ cannot have an inverse since it has zero determinant.

Example 11 *Find all values of x so that $M(x) = \begin{bmatrix} 5-x & 3 \\ 3 & 4-x \end{bmatrix}$ is singular. (Recall that a square matrix is singular exactly when it does not have an inverse.) We will compute $\det(M(x))$, and explore when it is zero. First, suppose that $5 - x = 0$. (That is, $x = 5$). Then $M(5) = \begin{bmatrix} 0 & 3 \\ 3 & -1 \end{bmatrix}$, and $\det(M(5)) = -\det\left(\begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}\right) = -(3)(3) = -9$, so $M(5)$ has an inverse. Now suppose that $5 - x \neq 0$, which allows us to divide by $5 - x$. Multiply the first row of $M(x)$ by $\frac{3}{5-x}$ and subtract it from the second row of $M(x)$. Thus,*

$$\begin{aligned} \det(M(x)) &= \det\left(\begin{bmatrix} 5-x & 3 \\ 0 & 4-x-\frac{9}{5-x} \end{bmatrix}\right) \\ &= (5-x)\left(4-x-\frac{9}{5-x}\right) \\ &= (5-x)(4-x)-9 \\ &= x^2-9x+11 \end{aligned}$$

$M(x)$ is singular exactly when $x^2 - 9x + 11 = 0$. That is, $x = \frac{1}{2}\left(9 \pm \sqrt{37}\right)$.

Exercise 12 *Let a and b be fixed real numbers. Find a quadratic polynomial $q(x)$ such that $N(x) = \begin{bmatrix} a-x & b \\ b & a-x \end{bmatrix}$ is singular exactly when x is a root of $q(x)$. What are the roots of $q(x)$? (Your answer will depend on a and b.)*

Exercise 13 *Let a and b be fixed real numbers. Find a quadratic polynomial $q(x)$ such that $P(x) = \begin{bmatrix} a-x & b \\ -b & a-x \end{bmatrix}$ is singular exactly when x is a root of $q(x)$. What are the roots of $q(x)$? How does your answer differ from your answer from the previous problem?*

Exercise 14 *Let a, b, c, d be fixed real numbers. Find a quadratic polynomial $q(x) = \alpha x^2 + \beta x + \gamma$ such that $R(x) = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$ is singular exactly when x is a root of $q(x)$. How are α, β and γ related to a, b, c, d?*

Exercise 15 *Let M be a square matrix. Suppose that there is a nonzero vector x such that $Mx = \mathbf{0}$. What, if anything, can we conclude about* $\det(M)$. *Explain.*

Exercise 16 *Let A be a square matrix and let x be a fixed scalar. We have seen that the set $\mathbf{V}_x = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = x\mathbf{v}\}$ is actually the null space of the matrix $(A - xI_n)$. Show that $\mathbf{V}_x$ contains a nonzero vector exactly when* $\det(A - xI_n) = 0$.

Exercise 17 *What restrictions on x guarantee that $\mathbf{V}_x = \{\mathbf{v} \in \mathbb{R}^2 : A\mathbf{v} = x\mathbf{v}\}$ contains a nonzero vector when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$? (Hint: Express your restrictions in terms of a polynomial equation.)*

Exercise 18 *Let A be an $n \times n$ matrix. Let x be a scalar. Show that* $\det(xI_n - A) = 0$ *if and only if* $\det(A - xI_n) = 0$. *Hint: First show that for any square matrix M,* $\det(M) = 0$ *if and only if* $\det(-M) = 0$.

2.2 Determinants and Transposition

Theorem 19 *Let A be an $n \times n$ matrix. Then* $\det(A) = \det(A^T)$.

Proof. We need to recall two facts about an invertible matrix: A matrix is invertible if and only if its transpose is invertible, and a matrix is invertible if and only if it can be written as a product of elementary matrices.

If A is not invertible, then A^T is not invertible, and hence, $\det(A) = 0$ and $\det(A^T) = 0$ both hold.

Suppose that A has an inverse. Then A can be written as a product of elementary matrices. That is, $A = F_1 F_2 \cdots F_{k-1} F_k$ for some elementary matrices Fj. Then

$$\begin{aligned} A^T &= (F_1 F_2 \cdots F_{k-1} F_k)^T \\ &= (F_k)^T (F_{k-1})^T \cdots (F_2)^T (F_1)^T. \end{aligned}$$

Recall that the transpose of an elementary matrix is also an elementary matrix. Further, recall that the transpose of a type I elementary matrix is a type I elementary matrix that uses the same scalar, that the transpose of a type II elementary matrix is the same type II elementary matrix, and that the transpose of a type III elementary matrix. Note that we can also write $A = (F_1F_2 \cdots F_{k-1}F_k)\, I_n$ and $A^T = \left((F_k)^T (F_{k-1})^T \cdots (F_2)^T (F_1)^T\right) I_n$. Thus when we compute the product of scalars for type I matrices, we use the same scalars for both A and A^T, and when we count row switches, we get the same number of row switches for both A and A^T. Since both A and A^T have $R = I_n$, Theorem 1 tells us that the determinants are equal. ■

Since the transpose of a lower triangular matrix is an upper triangular matrix, and vise versa, and since the determinant of both upper and lower triangular matrices depend only on the diagonal entries, which are not affected by transposition, we might not be surprised that transposition does not change the determinant. It is easy to check that the matrices U, V, K and L in the previous section all have the same determinants as their transposes.

2.3 Determinants and Products

Let A and B be $n \times n$ matrices. Recall that if A and B are invertible, then so is AB. Thus if both of $\det(A)$ and $\det(B)$ are nonzero, then $\det(AB)$ is nonzero. Also recall that if AB is not invertible, then at least one of A and B is not invertible. Thus if $\det(AB) = 0$, then at least one of $\det(A) = 0$ and $\det(B) = 0$ holds. That is, $\det(AB) = 0$ if and only if $\det(A)\det(B) = 0$. What happens when $\det(A)$, $\det(B)$ and $\det(AB)$ are all nonzero?

Theorem 20 *Let A and B be $n \times n$ matrices. Then* $\det(AB) = \det(A)\det(B)$.

Proof. In light of the preceding remarks, we only need to examine the case when A and B are invertible, so suppose that A and B are invertible matrices. Recall that an invertible matrix can be written as a product of elementary matrices. That is, there are elementary matrices F_j and G_j such that $A = F_1F_2 \cdots F_{k-1}F_k$ and $B = G_1G_2 \cdots G_{h-1}G_h$. Then $\det(A)$ is the product of the scalars from the type I elementary matrices among the F_j multiplied by (-1) raised to the number of type II elementary matrices that occur among the F_j. Similarly, $\det(B)$ is the product of the scalars from the type I elementary matrices among the G_j multiplied by (-1) raised to the number of type II elementary matrices that occur among the G_j. Now $AB = (F_1F_2 \cdots F_{k-1}F_k)\,(G_1G_2 \cdots G_{h-1}G_h) = F_1F_2 \cdots F_{k-1}F_kG_1G_2 \cdots G_{h-1}G_h$, and hence, $\det(AB)$ is the product of the scalars from the type I elementary matrices among the F_j and the G_j, all multiplied by (-1) raised to the number of type II elementary matrices that occur among the F_j and the G_j. That is,

$$\det(AB) = \left[(-1)^{\#\ of\ F\ and\ G\ of\ type\ II}\right] \prod_{F\ and\ G\ type\ I} (scalars)$$

$$= \left[(-1)^{(\#\ of\ F\ of\ type\ II)\ +\ (\#\ of\ G\ of\ type\ II)}\right] \prod_{F\ type\ I} (scalars) \prod_{G\ type\ I} (scalars)$$
$$= \left[(-1)^{\#\ of\ F\ of\ type\ II} \bullet (-1)^{\#\ of\ G\ of\ type\ II}\right] \prod_{F\ type\ I} (scalars) \prod_{G\ type\ I} (scalars)$$
$$= \left[(-1)^{\#\ of\ F\ of\ type\ II}\right] \prod_{F\ type\ I} (scalars) \bullet \left[(-1)^{\#\ of\ G\ of\ type\ II}\right] \prod_{G\ type\ I} (scalars)$$
$$= \det(A) \bullet \det(B).$$

■

Example 21 *From above, we know that* $A = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$ *has* $\det(A) = 0$, *and that* $B = M(5) = \begin{bmatrix} 0 & 3 \\ 3 & -1 \end{bmatrix}$ *has determinant* -9. *Observe that* $AB = \begin{bmatrix} 18 & 3 \\ 36 & 6 \end{bmatrix}$. *Since* AB *has echelon form* $\begin{bmatrix} 18 & 3 \\ 0 & 0 \end{bmatrix}$, $\det(AB) = 0 = \det(A)\det(B)$.

Example 22 *The matrix* $C = \begin{bmatrix} 1 & 7 \\ 4 & 3 \end{bmatrix}$ *is transformed to* $\begin{bmatrix} 1 & 7 \\ 0 & -25 \end{bmatrix}$ *by subtracting four copies of its first row from its second. Hence* $\det(C) = -25$. *Using* $B = M(5)$, *we see that* $BC = \begin{bmatrix} 12 & 9 \\ -1 & 18 \end{bmatrix}$. *The matrix* BC *is transformed into* $\begin{bmatrix} -1 & 18 \\ 0 & 225 \end{bmatrix}$ *by switching the two rows, and then adding twelve copies of the new first row to the second row. Thus* $\det(BC) = -(-1)(225) = 225 = \det(B)\det(C)$.

Example 23 *The matrix* $CB = \begin{bmatrix} 21 & -4 \\ 9 & 9 \end{bmatrix}$ *differs from* BC, *so* B *and* C *do not commute. Multiply the second row of* CB *by* $1/9$. *Switch the two rows. Subtract twenty-one copies of the new first row from the second. The result is* $\begin{bmatrix} 1 & 1 \\ 0 & -25 \end{bmatrix}$, *and hence,* $\det(CB) = (-1)^1 \frac{(1)(-25)}{1/9} = 225 = \det(BC)$.

The determinant is always a scalar. Consequently, $\det(A)\det(B) = \det(B)\det(A)$ for all A and B. Thus,

Corollary 24 *If* A *and* B *are* $n \times n$ *matrices, then* $\det(AB) = \det(BA)$, *even if* A *and* B *do not commute.*

Note that Theorem 20 says that for an elementary matrix E, $\det(EA) = \det(E)\det(A)$. It is easy to see that the determinant of a type I elementary matrix using the nonzero scalar c is exactly c since a type I elementary matrix is a diagonal matrix with all entries on the diagonal equal to 1 with the exception of a single diagonal entry of c. It is also easy to see that and that the determinant of a type III elementary matrix matrix is 1 since such a matrix is a triangular matrix with all diagonal entries equal to 1. It is difficult, in general to directly show that the determinant of a type II elementary matrix is -1.

Exercise 25 *Suppose that X, Y, Z are all $n \times n$ matrices. Use the product theorem to find an expression for* $\det(XYZ)$ *in terms of the determinants of X, Y and Z.*

Exercise 26 *Suppose that A is an $n \times n$ matrix. Find expressions for the determinants of A^2 and A^3.*

Exercise 27 *Suppose that A is an $n \times n$ matrix. Find an expression for the determinant of A^k that is valid for all positive integers k, and use mathematical induction to prove that your formula is valid.*

Exercise 28 *Suppose that A is a square matrix such that $A^2 = A$. What are the possible values for* $\det(A)$? *What are the possible values if you know that A has an inverse?*

Exercise 29 *Suppose that A is a square matrix such that $A^2 = I$. What are the possible values for* $\det(A)$?

Exercise 30 *For an $n \times n$ matrix A, we always have* $\det(AA) = (\det A)(\det A)$. *Does* $\det(A + A) = \det(A) + \det(A)$ *always hold?*

Exercise 31 *Let a and b be real numbers. Compute the determinants of A, B and $A + B$ where*

$$A = \begin{bmatrix} a+1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -b-1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Is there a simple equality relationship between $\det(A) + \det(B)$ *and* $\det(A + B)$*? This example illustrates a general fact, the relationship between* $\det(A) + \det(B)$ *and* $\det(A + B)$ *is very complicated..*

Exercise 32 *Compute the determinants of MN and NM for*

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } N = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Why don't your results violate the Corollary to Theorem 20?

Exercise 33 *Let A be an $n \times n$ matrix. Let x be a scalar. Show that if* $\det(A - xI_n) = 0$*, then* $\det(A^2 - x^2 I_n) = 0$. *Hint: $A^2 - x^2 I_n = (A - xI_n)(A + xI_n)$.*

2.4 Determinants and Inverses

We can use the product property of determinants to prove a useful relationship between the determinant of a matrix and the determinant of its inverse.

Theorem 34 *Let A be an invertible matrix. Then* $\det(A^{-1}) = (\det(A))^{-1}$.

Proof. Observe that $AA^{-1} = I_n$ for some n, and that $\det(I_n) = (1)^n = 1$. By the product rule for determinants, $\det(A)\det(A^{-1}) = 1$. ■

Without computing the inverse of the matrix C used in the previous examples, we know that $\det(C^{-1}) = (\det(C))^{-1} = (-225)^{-1} = \frac{-1}{225}$.

Later in the course, when we examine change of bases, especially in the context of eigenvectors, the following result will be very useful:

Theorem 35 *Let M be an $n \times n$ matrix. Let P be an invertible $n \times n$ matrix. Then* $\det(P^{-1}MP) = \det(M)$.
Proof. *See the exercises.* ■

Notice that this result does NOT say that $P^{-1}MP = M$, since such a statement is usually FALSE.

Exercise 36 *Suppose that $A^{-1} = A$. What possible value(s) could* $\det(A)$ *have?*

Exercise 37 *Recall that if E is a type II elementary matrix, then $E^{-1} = E$. What possible value(s) could E have? (Remember the property of type II matrices discussed in the very first section of these notes.)*

Exercise 38 *Suppose that $A^T = A^{-1}$. What possible value(s) could* $\det(A)$ *have?*

Exercise 39 *Prove Theorem 35.*

Exercise 40 *Suppose that A and B are 4×4 matrices with* $\det(A) = 3$ *and* $\det(B) = -1/2$. *Compute numerical values for the following four determinants:* $\det(2A^2)$, $\det(ABABA)$, $\det(A^3B^2)$, $\det(A^{-1}B)$. *Do NOT assume that A and B commute.*