

11 Fundamental Properties of Real Numbers

The symbol \mathbb{R} denotes the set of all real numbers. This set has two basic operations: multiplication and addition. This set and its operations satisfy 11 fundamental properties:

1. For all $c, d \in \mathbb{R}$, $c + d \in \mathbb{R}$.
(The set \mathbb{R} is *closed under addition*.)
2. For all $b, c, d \in \mathbb{R}$, $b + (c + d) = (b + c) + d$.
(Real number addition is *associative*.)
3. For all $c, d \in \mathbb{R}$, $c + d = d + c$.
(Real number addition is *commutative*.)
4. There is a zero element 0 in \mathbb{R} such that $0 + c = c$ for all $c \in \mathbb{R}$.
(\mathbb{R} contains an *additive identity element*.0.)
5. For each $c \in \mathbb{R}$, there is $-c \in \mathbb{R}$ such that $c + (-c) = 0$.
(Each number in \mathbb{R} has an *additive inverse* in \mathbb{R} .)
6. For all $c, d \in \mathbb{R}$, $cd \in \mathbb{R}$.
(The set \mathbb{R} is *closed under multiplication*.)
7. For all $b, c, d \in \mathbb{R}$, $b(cd) = (bc)d$.
(Real number multiplication is *associative*.)
8. For all $c, d \in \mathbb{R}$, $cd = dc$.
(Real number multiplication is *commutative*.)
9. For each $c \in \mathbb{R}$, $1c = c$.
(The set \mathbb{R} has a *multiplicative identity element*.1.)
10. For each **nonzero** $c \in \mathbb{R}$, there is $d \in \mathbb{R}$ such that $cd = 1$.
(Each nonzero number in \mathbb{R} has a *multiplicative inverse* in \mathbb{R} .)
11. For all $b, c, d \in \mathbb{R}$, $b(c + d) = bc + bd$.
(Real number multiplication and addition *distribute*.)

It can be shown that the additive and multiplicative identity elements are unique. That is, there is only one real number that satisfies Property 4, and there is only one real number that satisfies Property 9.

For each $c \in \mathbb{R}$, it can be shown that there is a unique additive inverse $-c$ in \mathbb{R} . Note that *subtraction* of real numbers is defined as addition of the additive inverse. That is, if $c, d \in \mathbb{R}$, then $c - d$ is defined by $c - d = c + (-d)$.

For each **nonzero** $c \in \mathbb{R}$, it can be shown that there is a unique, nonzero, multiplicative inverse in \mathbb{R} , which we will denote as c^{-1} . Note that *division* by a nonzero number is defined as multiplication by the multiplicative inverse. That is, if $c, d \in \mathbb{R}$ and d is nonzero, then c/d is defined by $c/d = c(d^{-1})$.

One can show, using only the 11 previous, listed properties above, that the following three additional properties are also true:

12. For all $c \in \mathbb{R}$, $(-1)c = -c$.

13. For all $c \in \mathbb{R}$, $0c = 0$.

14. For all $c, d \in \mathbb{R}$, if $cd = 0$ then at least one of $c = 0$ and $d = 0$ holds.

For example, here is the justification for Property 14. Suppose that c and d are real numbers with $cd = 0$. If $c = 0$, we are done. If $c \neq 0$, then we must show that $d = 0$. Start with $cd = 0$. Since $c \neq 0$, by Property 10, there is a real number c^{-1} so that $c^{-1}c = 1$. Multiply both sides of $cd = 0$ by c^{-1} as follows

$$\begin{aligned} cd &= 0 \\ c^{-1}(cd) &= c^{-1}0 \\ (c^{-1}c)d &= 0 \end{aligned}$$

where we have used Property 7 to transform the left hand side, and we have used Property 13 to transform the right hand side. Since $c^{-1}c = 1$, the left hand side is actually $1d$, and by Property 9, $1d = d$. Thus we have shown that $c \neq 0$ forces $d = 0$.

We will prove in class that Property 13 is a consequence of properties from Properties 1 – 11.

Homework Bonus Problem: Using Property 13 together with properties from Properties 1 – 11, prove Property 12.