The Fundamental Theorem of Algebra

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We have lots of experience with polynomials. Expressions such as $3x - 7$ and $5x^2 - 6x + 11$ are examples of polynomials. In fact, for such polynomials, we know a lot about the graphs. The graph of the linear polynomial $3x + 7$ is a line with slope 3 and intercept -7. The graph of the quadratic polynomial $5x^2 - 6x + 11$ is a parabola that opens up and has vertex at $(\frac{3}{5}, \frac{46}{5})$. An example of a more general polynomial is $5x^{13} - 0.367x^8 - \pi x^5 + 8.997x$. The *degree* of a polynomial is the biggest power that occurs with a nonzero coefficient. Thus the degrees of the polynomials above, in order of appearance, are 1, 2 and 13. A generic n^{th} degree polynomial is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the leading coefficient, a_n is nonzero. (If $a_n = 0$ then the biggest power of x with a nonzero coefficient, and hence the degree, would be less than n.) The numbers $a_0, a_1, \ldots, a_n$ are constants chosen from an appropriate set such as the integers, the rational numbers, the real numbers or even the complex numbers. We call the numbers $a_0, a_1, \ldots, a_n$ *coefficients* of the corresponding powers of x. Thus a_n is the coefficient of x^n, a_2 is the coefficient of x^2, and a_1 is the coefficient of $x = x^1$. The number a_0 is called the *constant term* in the polynomial $p(x)$.

What about roots of polynomials? Recall that the number r is a *root* for the polynomial $p(x)$ means that $p(r) = 0$. For the first degree polynomial $p(x) = mx + b$ where $m \neq 0$, we have a line with nonzero slope, and consequently, it crosses the x-axis exactly once, at $r = \frac{-b}{m}$. Notice that $p(x) = mx + b = m(x - r)$. For the second degree polynomial $p(x) = ax^2 + bx + c$, we know from the quadratic formula that there are three cases: two different real roots when $b^2 - 4ac > 0$ (call them r_1 and r_2), two copies of the same root when $b^2 - 4ac = 0$ (so $r_1 = r_2$), and no real roots when $b^2 - 4ac < 0$. In fact, when $b^2 - 4ac < 0$, there are two nonreal complex roots that form a conjugate pair:

$$r_1 = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} \quad \textit{and} \quad r_2 = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}$$

where i is the imaginary number that satisfies $i^2 = -1$. It can be shown by substitution and multiplication that $ax^2 + bx + c = a(x - r_1)(x - r_2)$ in all three cases. Note that when $r_1 = r_2$, we have $p(x) = ax^2 + bx + c = a(x - r_1)^2$; we call r_1 a *double root* for $p(x)$, or we say that r_1 is a root of $p(x)$ with *multiplicity* 2.

Notice that roots are closely associated with factoring. Specifically, $p(r) = 0$ means that there is a polynomial $q(x)$ such that $p(x) = (x - r)q(x)$. For an n^{th} degree polynomial $p(x)$, we say that r is a root with multiplicity k if k is the biggest number such that there is a polynomial $q(x)$ satisfying

$$p(x) = (x - r)^k q(x).$$

Notice that every root has multiplicity of at least 1. As an example, $r = 5$ is a root of $13x^3(x-5)^7(x^2+9)^2$ with multiplicity 7; and 0 is a root of multiplicity 3 (since $x^3 = (x-0)^3$). Since $x^2+9 = (x+3i)(x-3i)$, the conjugate roots $+3i$ and $-3i$ both have multiplicity 2. Counting multiplicities, the polynomial $13x^3(x-5)^7(x^2+9)^2$ has a total of 14 roots. Notice that if we multiplied out this factored polynomial that the leading term would be $13x^{14}$, so the polynomial has degree 14. Is this a coincidence? The next results tells us that it is NOT a coincidence.

Theorem 1 ***(Fundamental Theorem of Algebra for Real Polynomials)*** *Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with integer, rational or real coefficients such that $a_n \neq 0$. Then $p(x)$ has a total of exactly n roots counting multiplicities of all real and nonreal complex roots. Further, any nonreal complex roots occur in conjugate pairs. Finally, if we list the roots as $r_1, r_2, \ldots, r_n$, then $p(x)$ factors as*

$$p(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$

Note that if some root of $p(x)$ has multiplicity bigger than 1, then two or more of the listed roots will be the same number. Also, if some r_j is nonreal, then another root in the list must be its complex conjugate. Indeed, the total number of nonreal complex roots must be even.

From the previous theorem, $p(x) = 47x^5 - 11x^3 + \frac{\pi}{3}x^2 - \sqrt{7}$ must have a total of 5 roots. Further, since any nonreal complex roots must occur in conjugate pairs, exactly one of the following must be true about the five roots: all five are real, exactly three are real and there is one nonreal, complex conjugate pair, or exactly one is real and there are two complex conjugate pairs. (In fact, there is only one real root, so the other four roots form two pairs of nonreal, complex conjugate roots.)

In fact, there is a somewhat more general result:

Theorem 2 ***(Fundamental Theorem of Algebra - complex version)*** *Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real or complex coefficients such that $a_n \neq 0$. Then $p(x)$ has a total of exactly n roots counting multiplicities of all real and nonreal complex roots. Further, if we list the roots as $r_1, r_2, \ldots, r_n$, then $p(x)$ factors as*

$$p(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$

Note that if some root of $p(x)$ has multiplicity bigger than 1, then two or more of the listed roots will be the same number. If one or more of the coefficients in $p(x)$ is a nonreal complex number, then we no longer know a priori that nonreal complex roots have to occur in conjugate pairs.