The Geometric Series

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The Geometric Series – Finite Number of Summands

The geometric series is a sum in which each summand is obtained as a common multiple of the previous summand. Thus,

$$30 + 300 + 3000 + 30000$$

is a geometric sum where the common multiple is 10. Note that we can always factor out the first term in a series,

$$\begin{aligned} 30 + 300 + 3000 + 30000 &= 30\,(1 + 10 + 100 + 1000) \\ &= 30\left(1 + 10^1 + 10^2 + 10^3\right) \end{aligned}$$

Consequently, it is convenient to concentrate our attention on the geometric series in which the first summand is 1. Additionally, there is nothing magical about 10; we could just as easily had powers of 2 or 1/2 or even π. That is, we could have had a sum of powers of *any* real number. For convenience, let r denote an unspecified real number as the common multiple. Further, there is no reason why we should only have 4 summands, thus we can let the last summand be the $(n+1)^{st}$ summand, r^n. Then we have

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

as the geometric series consisting of exactly $n+1$ summands. Let us denote this series with $n+1$ terms by G_n, with the understanding that $G_0 = 1$. A natural question is, can we find a formula (in terms of n and r) that determines the sum more conveniently than simply adding the terms one after another? That is, we want to write G_n as a nice function of r and n.

$$G_n = 1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Note that

$$\begin{aligned} 1 + rG_n &= 1 + r\left(1 + r + r^2 + r^3 + \cdots r^{n-1} + r^n\right) \\ &= 1 + r + r^2 + r^3 + r^4 + \cdots r^n + r^{n+1} \\ &= \left(1 + r + r^2 + r^3 + r^4 + \cdots r^n\right) + r^{n+1} \\ &= G_n + r^{n+1} \end{aligned}$$

That is,

$$1 + rG_n = G_n + r^{n+1}.$$

Then

$$\begin{aligned} 1 - r^{n+1} &= G_n - rG_n \\ 1 - r^{n+1} &= (1-r)G_n. \end{aligned}$$

Since we want to solve for G_n, we would like to divide by $1-r$. Before we can divide, however, we have to be sure that $1-r \neq 0$, which is to say, that $r \neq 1$. So we have

$$G_n = \frac{1-r^{n+1}}{1-r} \text{ when } r \neq 1. \qquad (*)$$

What happens when $r = 1$? When $r = 1$, we have $r^k = 1$ for each positive integer k, so G_n is the sum of $n+1$ copies of 1.

Let us return to our original example,

$$\begin{aligned} 30 + 300 + 3000 + 30000 &= 30\left(1 + 10^1 + 10^2 + 10^3\right) \\ &= 30\frac{1-10^{3+1}}{1-10} = 30\frac{-9999}{-9} \\ &= 30(1111) = 33330. \end{aligned}$$

As another example, suppose that $r = -1$. Then

$$\begin{aligned} G_n &= 1 + (-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^n \\ &= \frac{1-(-1)^{n+1}}{1-(-1)} = \frac{1-(-1)^{n+1}}{2} \end{aligned}$$

Since $(-1)^{n+1} = 1$ when n is odd, and $(-1)^{n+1} = -1$ when n is even, it follows that $G_n = 0$ when n is odd, and $G_n = 1$ when n is even.

The Geometric Series – Infinitely Many Summands

What happens to the series as the number of summands grows larger and larger? As we have just seen, when $r = -1$, as n grows larger, G_n flips back and forth between 0 and -1, depending on whether n is odd or n is even. When $r = 10$, the series is

$$G_n = 1 + 10^1 + 10^2 + 10^3 + \cdots + 10^n = 1111\cdots1111 \ (n+1 \text{ digits})$$

so as n grows towards infinity, G_n grows towards positive infinity. On the other hand, when $r = 1/10$,

$$\begin{aligned} G_n &= 1 + (1/10)^1 + (1/10)^2 + (1/10)^3 + \cdots + (1/10)^n \\ &= 1 + (0.1)^1 + (0.1)^2 + (0.1)^3 + \cdots + (0.1)^n \\ &= 1 + 0.1 + 0.01 + 0.001 + \cdots + 0.00\cdots01 \\ &= 1.111\cdots1111 \ (n \text{ decimal places to the right of } 0) \end{aligned}$$

As n grows towards infinity, G_n grows towards $1 + 1/9$. More generally, we are interested in the limit of G_n as $n \to \infty$. That is, when is there a real number G so that

$$G = \lim_{n\to\infty} G_n \ ?$$

When $r = 1$, $G_n = n + 1$, so $\lim_{n\to\infty} G_n = \lim_{n\to\infty}(n + 1) = +\infty$, and thus the series diverges (to plus infinity). When $r = -1$, we have seen that G_n flips back and forth between 0 and -1 as $n \to \infty$, so $\lim_{n\to\infty} G_n$ cannot exist, and thus the series diverges (by oscillation). More generally, when $r \neq 1$, we can use formula $(*)$,

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1 - r}\right) = \frac{1}{1-r} \lim_{n\to\infty} \left(1 - r^{n+1}\right) = \frac{1}{1-r}\left(1 - r \cdot \lim_{n\to\infty} r^n\right)$$

Thus the behavior of $\lim_{n\to\infty} G_n$ is determined by $\lim_{n\to\infty} r^n$. It is easy to see that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$, and that $\lim_{n\to\infty} |r^n| = +\infty$ when $|r| > 1$. Thus when $|r| < 1$,

$$\lim_{n\to\infty} G_n = \frac{1}{1-r}\left(1 - r \cdot \lim_{n\to\infty} r^n\right) = \frac{1}{1-r}(1 - r \cdot 0) = \frac{1}{1-r}.$$

When $|r| > 1$, $= 1$, $\lim_{n\to\infty} G_n$ does not exist. Finally, when $|r| = 1$, and hence, $r = 1$ or $r = -1$, we have seen that $\lim_{n\to\infty} G_n$ does not exist.
Summarizing,

$$G = \lim_{n\to\infty} G_n = \frac{1}{1-r} \text{ when } |r| < 1$$

$$G = \lim_{n\to\infty} G_n \text{ does NOT exist when } |r| \geq 1.$$

Thus, for example, when $r = 1/2$, the series

$$\begin{aligned} G &= 1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots + 1/2^n + \cdots \\ &= 1 + (1/2)^1 + (1/2)^2 + (1/2)^3 + \cdots + (1/2)^n + \cdots \\ &= \frac{1}{1 - \frac{1}{2}} = 2 \end{aligned}$$

and when $r = -1/2$, the series

$$\begin{aligned} G &= 1 - 1/2 + 1/4 - 1/8 + 1/16 - + \cdots + (-1)^n/2^n + \cdots \\ &= 1 + (-1/2)^1 + (-1/2)^2 + (-1/2)^3 + \cdots + (-1/2)^n + \cdots \\ &= \frac{1}{1 - \frac{-1}{2}} = 2/3 \end{aligned}$$

In fact, if we switch from working with real numbers to working with complex numbers, and if we replace the absolute value of a real number with the magnitude of a complex number, then for r *complex*, the series converges to $\frac{1}{1-r}$ exactly when the magnitude of r, denoted $|r|$, satisfies $|r| < 1$.

The Function $G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$

The function $(1-x)^{-1}$ is defined for all $x \neq 1$. From the preceding section, we see that **if we restrict** x **to** $|x| < 1$, then

$$G(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

Thus we have expressed the function $(1-x)^{-1}$ in terms of an infinite series of powers of x, specifically, when $|x| < 1$, $(1-x)^{-1}$has the same value as the geometric series $G(x)$.

While the geometric series by itself is interesting, its usefulness is that we can use the geometric series to find formulae for a variety of related series obtained by algebra and by calculus.

New Series via Algebra and via Substitution

Using algebra, we can connect new power series with new functions. Thus

$$\begin{aligned} T(x) &= 8 + 3x^2 + 3x^3 + \cdots + 3x^n + \cdots \\ &= 8 + 3x^2\left(1 + x + x^2 + \cdots + x^n + \cdots\right) \\ &= 8 + 3x^2 \cdot G(x) \\ &= 8 + 3x^2 \cdot \frac{1}{1-x} = 8 + \frac{3x^2}{1-x} \end{aligned}$$

for all x satisfying $|x| < 1$. As another example,

$$\begin{aligned} U(x) &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \\ &= 1 + (-x)^1 + (-x)^2 + (-x)^3 + \cdots + (-x)^n + \cdots \\ &= G(-x) = \frac{1}{1-(-x)} = \frac{1}{1+x} \end{aligned}$$

again, for all x satisfying $|x| < 1$.

Suppose we want to understand the series

$$V(u) = 1 - u^2 + u^4 - u^6 + \cdots + (-1)^n u^{2n} + \cdots$$

If we substitute $x = -u^2$ into $G(x)$, we get

$$\begin{aligned} V(u) &= 1 - u^2 + u^4 - u^6 + \cdots + (-1)^n u^{2n} + \cdots \\ &= 1 + (-u^2) + \left(-u^2\right)^2 + \left(-u^2\right)^3 + \cdots + (-u^2)^n + \cdots \\ &= G(-u^2) = \frac{1}{1-(-u^2)} = \frac{1}{1+u^2} \end{aligned}$$

That is,

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \cdots + (-1)^n u^{2n} + \cdots$$

exactly when $\left|-u^2\right| < 1$. That is, exactly when $|u| < 1$.

We can combine substitutions and algebra. Suppose that

$$h(x) = \frac{x}{4-5x}.$$

Rewriting $h(x)$ as

$$h(x) = \frac{x}{4} \cdot \frac{1}{1-(5x/4)} = \frac{x}{4} \cdot G\left(\frac{5x}{4}\right)$$

so we have

$$\begin{aligned} h(x) &= \frac{x}{4}\left(1 + \frac{5x}{4} + \left(\frac{5x}{4}\right)^2 + \left(\frac{5x}{4}\right)^3 + \cdots + \left(\frac{5x}{4}\right)^n + \cdots\right) \\ &= \frac{x}{4}\left(1 + \frac{5}{4}x + \frac{5^2}{4^2}x^2 + \frac{5^3}{4^3}x^3 + \cdots + \frac{5^n}{4^n}x^n + \cdots\right) \\ &= \frac{x}{4} + \frac{5}{4^2}x^2 + \frac{5^2}{4^3}x^3 + \frac{5^3}{4^4}x^4 + \cdots + \frac{5^n}{4^{n+1}}x^{n+1} + \cdots \end{aligned}$$

which converges exactly when $|5x/4| < 1$. That is, the series converges to $h(x)$ exactly when $|x| < 4/5$.

Some functions that do not appear to be related to $(1-x)^{-1}$ can be creatively rewritten. Thus

$$\frac{1}{x} = \frac{1}{1-(1-x)} = G(1-x) = G\left(-(x-1)\right)$$

So we obtain

$$\begin{aligned} \frac{1}{x} &= 1 + (-(x-1)) + (-(x-1))^2 + (-(x-1))^3 + \cdots + (-(x-1))^n + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots \end{aligned}$$

This series converges exactly when $|r| < 1$, which is to say, exactly when $|x-1| < 1$.

As a final example in this section, a series for

$$g(x) = \frac{1}{4-5x^3} = \frac{1}{4} \cdot \frac{1}{1-(5x^3/4)}$$

can be obtained by substituting $\frac{5}{4}x^3$ into the series $G(x)$:

$$\begin{aligned} \frac{1}{4-5x^3} &= \frac{1}{4} \cdot G\left(\frac{5x^3}{4}\right) = \frac{1}{4}\left(1 + \frac{5}{4}x^3 + \left(\frac{5}{4}x^3\right)^2 + \left(\frac{5}{4}x^3\right)^3 + \cdots + \left(\frac{5}{4}x^3\right)^n + \cdots\right) \\ &= \frac{1}{4} + \frac{5}{4^2}x^3 + \frac{5^2}{4^3}x^6 + \frac{5^3}{4^4}x^9 + \cdots + \frac{5^n}{4^{n+1}}x^{3n} + \cdots \end{aligned}$$

which converges exactly when $\left|\frac{5}{4}x^3\right| < 1$. Rearranging, this gives $\left|x^3\right| < \frac{4}{5}$, and hence, the series for $(4-5x^3)^{-1}$ converges exactly when

$$|x| < \sqrt[3]{\frac{4}{5}}.$$

New Series via Differentiation

How can calculus be used to develop new series from old ones? Let us first examine differentiation. A very important theorem tells us that we can differentiate series:

Theorem 1 ***(Derivatives of Series)*** *Suppose that the series $T(x)$ given by*

$$T(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots$$

converges for $|x| < R$ where R is some positive real number, and that the series diverges when $|x| > R$. Then $T(x)$ can be differentiated according to the sum rule, so

$$T'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots$$

$$T''(x) = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \cdots + n(n-1)a_nx^{n-2} + \cdots$$

and so on for higher order derivatives. Further, each derivative of $T(x)$ converges for $|x| < R$ and diverges for $|x| > R$.

Note that the theorem does NOT tell us what happens when $|x| = R$. Analyzing the cases when $|x| = R$ is often quite difficult, and is beyond our knowledge.

Returning to $G(x)$ from the previous sections,

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n + \cdots$$

which converges exactly when $|x| < 1$, (so $R = 1$ in the theorem above). We see that

$$G'(x) = 0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots + nx^{n-1} + \cdots$$

Since

$$G(x) = \frac{1}{1-x} = (1-x)^{-1}$$

it follows from the chain rule that

$$G'(x) = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}.$$

Thus,

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}$$

for $|x| < 1$.

If we differentiate both sides a second time, we obtain

$$G''(x) = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots + n(n-1)x^{n-2} + \cdots = \frac{2}{(1-x)^3}$$

And if we differentiate both sides a third time, we obtain

$$G'''(x) = 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2x + 5 \cdot 4 \cdot 3x^2 + \cdots + n(n-1)(n-2)x^{n-3} + \cdots = \frac{2 \cdot 3}{(1-x)^4}$$

Each of these new series converges for $|x| < 1$ and diverges for $|x| > 1$. (What happens when $|x| = 1$ is not clear.)

We can combine differentiation with our techniques of algebra and substitution. For example,

$$\begin{aligned}
\frac{1}{(1+8x^2)^3} &= \frac{1}{\left(1-(-8x^2)\right)^3} = \frac{1}{2}G''(-8x^2) \\
&= \frac{1}{2}\left(2 \cdot 1 + 3 \cdot 2(-8x^2) + 4 \cdot 3(-8x^2)^2 + 5 \cdot 4(-8x^2)^3 + \cdots + n(n-1)(-8x^2)^{n-2} + \cdots\right) \\
&= 1 - 24x^2 + 384x^4 - 5120x^6 + \cdots + \frac{1}{2}(-1)^{n-2}8^{n-2}n(n-1)(n-2)x^{2n-4} + \cdots
\end{aligned}$$

which converges for $\left|-8x^2\right| < 1$ and diverges for $\left|-8x^2\right| > 1$. That is to say, the series converges when

$$|x| < \frac{1}{\sqrt{8}}$$

and diverges when

$$|x| > \frac{1}{\sqrt{8}}.$$

New Series via Integration

Let us now turn to integration. Again, we will need a theorem:

Theorem 2 ***(Integrals of Series)*** *Suppose that the series $T(x)$ given by*

$$T(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots$$

converges for $|x| < R$ where R is some positive real number, and that the series diverges when $|x| > R$. Then $T(x)$ can be integrated according to the sum rule, so

$$\int T(x)\ dx = K + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + a_4\frac{x^5}{5} + \cdots + a_n\frac{x^{n+1}}{n+1} + \cdots$$

for some constant K. Further,

$$I(x) = \int_0^x T(t)\ dt = a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + a_4\frac{x^5}{5} + \cdots + a_n\frac{x^{n+1}}{n+1} + \cdots$$

converges for $|x| < R$ and diverges for $|x| > R$.

Note that the theorem does NOT tell us what happens when $|x| = R$. Analyzing the cases when $|x| = R$ is often quite difficult, and the techniques needed to do so are often beyond our knowledge level in this course.

Returning to $G(x)$ from the previous sections,

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

which converges exactly when $|x| < 1$, (so $R = 1$ in the theorem above). We see that

$$\int G(x)\ dx = K + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots + \frac{x^{n+1}}{n+1} + \cdots$$

Alternatively,

$$\int_0^x G(t)\ dt = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots + \frac{x^{n+1}}{n+1} + \cdots$$

But notice that

$$\begin{aligned} \int_0^x G(t)\ dt &= \int_0^x \frac{1}{1-t}\ dt = -\ln(1-t)|_0^x = -\ln(1-x) + \ln(1+0) \\ &= -\ln(1-x) \end{aligned}$$

Multiplying both integral expressions by -1 yields

$$\begin{aligned} \ln(1-x) &= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots + \frac{x^{n+1}}{n+1} + \cdots\right) \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots - \frac{x^{n+1}}{n+1} - \cdots \end{aligned}$$

which converges for $|x| < 1$ and diverges for $|x| > 1$.

Substituting $x = -u$ into the previous result gives us

$$\begin{aligned} \ln(1+u) &= \ln\left(1 - (-u)\right) \\ &= -(-u) - \frac{(-u)^2}{2} - \frac{(-u)^3}{3} - \frac{(-u)^4}{4} - \frac{(-u)^5}{5} - \cdots - \frac{(-u)^{n+1}}{n+1} - \cdots \\ &= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \cdots + (-1)^{n+2}\frac{u^{n+1}}{n+1} - \cdots \end{aligned}$$

which converges for $|-u| < 1$ and diverges for $|-u| > 1$.

As another example, consider the series $V(u)$ given on page 4.

$$V(u) = 1 - u^2 + u^4 - u^6 + \cdots + (-1)^n u^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n u^{2n} = \frac{1}{1+u^2}$$

which converges exactly when $|u| < 1$. Since

$$\arctan(t) = \int_0^t \frac{1}{1+u^2}\ du,$$

it follows that for $|t| < 1$,

$$\arctan(t) = \int_0^t \sum_{n=0}^{\infty} (-1)^n u^{2n}\, du = \sum_{n=0}^{\infty} (-1)^n \int_0^t u^{2n}\, du = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\,.$$

This series, which must converge when $|t| < 1$ and diverge when $|t| > 1$, actually converges when $t = 1$. Since $\arctan(1) = \frac{\pi}{4}$,

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

As an approximation,

$$\pi \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}\right) = 3.0418...$$

which is not very good. The fact that $t = 1$ sits on the boundary between the t values where the series converges and the t values where the series diverges means that it takes many terms in the series to get a good approximation. It can be shown that we need at about 250,000 terms (!) to get the approximation $\pi \approx 3.14159$. In contrast, we could have used the fact that $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ and the same power series with $t = \frac{1}{\sqrt{3}}$ to get $\pi \approx 3.14159$ using only ten terms! Why the difference? $t = \frac{1}{\sqrt{3}} \approx 0.577$, which is much closer to the center of the interval of convergence ($t = 0$) than $t = 1$ is. This illustrates a general fact about series: The closer the variable is to the center of the interval of convergence, the faster the series converges, or equivalently, the fewer the number of terms needed for a good approximation.

Geometric Series Applied to Repeating Decimal Numbers

The numbers $0.333333\ldots$, $4.50621212121\ldots$ and $-81.5926130713071307\ldots$ are all called *repeating decimal numbers* because past a finite number of decimal places, each settles into a pattern consisting of a finite string of digits that repeats infinitely often. Given any repeating decimal number, we can always multiply it by a power of ten so that the result has its repeating pattern starting at the decimal point. Consider the three numbers at the start of this problem: the first number is already in this form (multiply it by $10^0 = 1$) with string "3", the second number should be multiplied by 10^3 and has string "21"; and the third number should be multiplied by 10^4 and has string "1307". Observe that

$$\begin{aligned}
0.333333\ldots &= 0.3 + 0.03 + 0.003 + 0.0003 + \cdots \\
&= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots \\
&= 3 \cdot \frac{1}{10} + 3 \cdot \frac{1}{10^2} + 3 \cdot \frac{1}{10^3} + 3 \cdot \frac{1}{10^4} + \cdots \\
&= 3\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right) \\
&= 3 \cdot \frac{1}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right) \\
&= \frac{3}{10} \cdot G\left(\frac{1}{10}\right) = \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} \\
&= \frac{3}{10 - 1} = \frac{3}{9} = \frac{1}{3}
\end{aligned}$$

This is perhaps the most complicated justification you have ever seen for why $0.333333\ldots = \frac{1}{3}$. Why did we work so hard? So that we might understand how to handle more complicated cases. Let us look at the second number in our list $a = 4.50621212121\ldots$. Since $10^3a = 4506.21212121\ldots$, we will work with $10^3a - 4506 = 0.21212121\ldots$. Observe that

$$\begin{aligned}
0.21212121\ldots &= 0.21 + 0.0021 + 0.000021 + 0.00000021 + \cdots \\
&= \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \frac{21}{100000000} + \cdots \\
&= 21 \cdot \frac{1}{100} + 21 \cdot \frac{1}{100^2} + 21 \cdot \frac{1}{100^3} + 21 \cdot \frac{1}{100^4} + \cdots \\
&= 21\left(\frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \frac{1}{100^4} + \cdots\right) \\
&= 21 \cdot \frac{1}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots\right) \\
&= \frac{21}{100} \cdot G\left(\frac{1}{100}\right) = \frac{21}{100} \cdot \frac{1}{1 - \frac{1}{100}} \\
&= \frac{21}{100 - 1} = \frac{21}{99} = \frac{7}{33}
\end{aligned}$$

Then $1000a - 4506 = \frac{7}{33}$, so

$$a = \frac{4506 + \frac{7}{33}}{1000} = \frac{148705}{33000}$$

A rational number is a real number that can be written as the quotient of two integers. Using the approach in this section and the ideas in Exercises 10 and 11 below, one can show that *every* repeating decimal number is a rational number.

In fact, every rational number can be written either as a decimal number with a finite number of nonzero digits, or else as a repeating decimal number. There are many, many real numbers that require infinitely many nonzero digits when written in decimal form but that are not rational numbers, and hence, cannot be written as repeating decimal numbers. Some famous examples are $\sqrt{2}$, π and e. One need not hunt for such exotic numbers to find an irrational number. The number $0.101001000100001000001\ldots$ is an irrational number with a simple pattern (but not a repeating pattern).

Exercises

1. In this problem, $G(x)$ is the geometric series with infinitely many terms, and $G_n(x)$ is the finite geometric series whose last term is x^n.
   (a) Using formula $(*)$, write the formula for the error due to approximating $G(x) = \frac{1}{1-x}$ by $G_n(x)$, that is, write the formula for the difference $G(x) - G_n(x)$ as a simple fraction.
   (b) When $n = 5$, how big is the error if $|x| \leq 0.9$? If $|x| \leq 0.1$?
   (c) When $n = 10$, how big is the error if $|x| \leq 0.9$? If $|x| \leq 0.1$?

2. Find a series for $f(x)$ in terms of powers of x where
$$f(x) = \frac{2x^3}{x^4 - 3}$$
   Be sure to indicate for which values of x the series converges, and for which values of x the series diverges.

3. Consider the sum
$$8x^3 - 16x^4 + 32x^5 - 64x^6 + \cdots$$
   (a) What is the coefficient of x^n in this series?
   (b) Find a formula for the series.
   (c) For what values of x does the series converge to the value given by your formula?

4. Find the series for
$$\frac{1}{(1-u)^4}$$
   and then use your result to find the series for
$$\frac{5x^3}{(1+3x^2)^4}$$
   and indicate for which values of x the series must converge, and for which values of x the series must diverge.

5. Find a series for

$$H(x) = \int_0^x \ln(1+t^3)\ dt$$

and indicate for which values of x the series must converge, and for which values of x the series must diverge. (Hint: First, find a series for $\ln(1+t^3)$ starting with the series for $\ln(1+u)$ developed on page 8.)

6. If $H(x)$ is the function that you found in exercise 5, find a series for

$$\int_0^x H(t)\ dt$$

7. We used the series for the arctangent function to approximate π twice, once using the fact that $\arctan(1) = \frac{\pi}{4}$ and once using the fact that $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$. Can we use the series a third time to approximate π using the fact that $\arctan(\sqrt{3}) = \frac{\pi}{3}$? Why or why not?

8. On page 8, we developed a series for $\ln(1-x)$. Suppose we want to approximate $\ln(9)$.

   (a) Explain why we cannot simply let $x = -8$ in our series.

   (b) Explain why an approximation for $\ln(1/9)$ could be used to approximate $\ln(9)$.

   (c) What value of x should we use in our series to compute $\ln(1/9)$? Is the series valid for that choice of x?

   (d) What is the relationship between $\ln(9)$ and $\ln(1/3)$?

   (e) What value of x should we use in our series to compute $\ln(1/3)$? Is the series valid for that choice of x?

   (f) Using the first five terms from the series, use your x values from parts (c) and (e) to approximate $\ln(1/9)$ and $\ln(1/3)$, and then use those approximations to approximate $\ln(9)$. Which x value leads to a better approximation of $\ln(9)$? Which x value is closer to the center of the interval of convergence?

9. End points of intervals of convergence are tricky. Look at the series for $\ln(1+x)$.

   (a) Look at approximations when $x = 1$. That is, look at what happens when you use only the first n terms for $n = 1, 2, 3, 4, 5, 10, 20$. Do the numbers appear to converge to $\ln(2)$?

   (b) Look at approximations when $x = -1$. That is, look at what happens when you use only the first n terms for $n = 1, 2, 3, 4, 5, 10, 20$. We know that $\ln(0)$ is undefined because as $x \to 0^+$, $\ln(x) \to -\infty$. Do your approximations appear to head towards $-\infty$?

(c) We know that $\ln(e) = 1$. What happens when you try to plug in $x = e - 1$ into your series for $\ln(1 + x)$? Look at the size of the n^{th} term as n gets bigger and bigger. Why isn't it surprising that the series is badly behaved when $x = e - 1$. (Hint: What is the numerical value of $e - 1$?)

10. Let $s = -81.5926130713071307\ldots$. Use the technique presented in the last two examples to express s as a ratio of two integers. (Hint: It might be easier to work with $-s$.)

11. Let us reconsider $a = 4.50621212121\ldots$ from the last example. Multiply a by 10^4 to obtain $45062.12121212\ldots$, and then subtract 45062 to obtain $0.12121212\ldots$. Apply our technique for finding a rational number for a repeating decimal number to $012121212\ldots$,and then solve for a. Show that you get the same rational number for a that we found previously.

12. Suppose that r is a positive repeating decimal number. Explain why there must be positive integers k and m so that $10^k r - m$ is a repeating decimal number of the form $0.string\,string\,string\ldots$ where $'string'$ denotes a repeating block of digits.

13. Suppose that r is a repeating decimal number of the form

$$0.string\ string\ string\ string\ \ldots$$

where $'string'$ denotes a repeating block consisting of 4 digits. That is, $'string' = abcd$ where $a, b, c, d \in \{0, 1, 2, 3, \ldots, 9\}$. Using the techniques employed in the last two examples, write r as the ratio of two integers. Your answer should involve the string $'abcd'$, .which can be viewed as a four digit integer.

14. We usually think about numbers base 10, but one can work with "decimal" numbers in other bases. For example, 0.11 base 2 is actually $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, while 0.11 base 3 is actually $\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$. For any whole number base (base 2, base 35,...), every repeating "decimal" is actually a rational number. Convert the following "decimal" numbers into their rational equivalents.

    (a) $0.111111\ldots$ base 3

    (b) $0.111111\ldots$ base 2

    (c) $0.121212\ldots$ base 3

    (d) $0.222222\ldots$ base 3

    (e) Are the results of questions (b) and (d) really surprising? What would be the analogous decimal string base 10 ?

    (f) Write $1/3$ as a "decimal" in base 2.

    (g) Write $1/3$ as a "decimal" in base 3.

    (h) Write $1/3$ as a "decimal" in base 6.