Solving a Linear System – An Extended Example

Suppose that the linear system $Ax = b$ consists of five equations in seven variables. Then the theory of linear systems tells us that either $Ax = b$ is inconsistent (no solutions), or else there are infinitely many solutions depending on at least $7 - 5 = 2$ independent, arbitrary parameters.

Suppose that $Ax = b$ has been transformed by the Gauss Jordan process to the augmented array

$$\left[\begin{array}{ccccccc|c} 1 & -3 & 0 & 0 & 4 & 0 & 7 & 9 \\ 0 & 0 & 1 & 0 & 5 & 0 & 3 & 8 \\ 0 & 0 & 0 & 1 & 6 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a row consisting entirely of zeros - both on the coefficient side *and* on the right-hand side, one of the original five equations was redundant. Since there are four leading nonzeros, there will be four leading variables (x_1, x_3, x_4, x_6). There must then be $7 - 4 = 3$ nonleading variables. We must solve for the leading variables in terms of the right-hand side and the nonleading variables (x_2, x_5, x_7). Working from the bottom upwards, we obtain:

$$\begin{aligned} x_6 &= 11 \\ x_4 &= 1 - 6x_5 + 5x_7 \\ x_3 &= 8 - 5x_5 - 3x_7 \\ x_1 &= 9 + 3x_2 - 4x_5 - 7x_7 \\ & \quad x_2, x_5, x_7 \text{ arbitrary} \end{aligned}$$

Apparently, there are infinitely many solutions to $Ax = b$ since we need to specify the values for each of three arbitrary parameters to completely identify a solution.

We can re-write the solution as a column vector in $\mathbf{R}^7$. If we substitute our expressions for the leading variables in terms of the right-hand side and the nonleading variables, we obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 9 + 3x_2 - 4x_5 - 7x_7 \\ x_2 \\ 8 - 5x_5 - 3x_7 \\ 1 - 6x_5 + 5x_7 \\ x_5 \\ 11 \\ x_7 \end{bmatrix} = \begin{bmatrix} 9 + 3x_2 - 4x_5 - 7x_7 \\ 0 + 1x_2 + 0x_5 + 0x_7 \\ 8 + 0x_2 - 5x_5 - 3x_7 \\ 1 + 0x_2 - 6x_5 + 5x_7 \\ 0 + 0x_2 + 1x_5 + 0x_7 \\ 11 + 0x_2 + 0x_5 + 0x_7 \\ 0 + 0x_2 + 0x_5 + 1x_7 \end{bmatrix}$$

Extracting the nonleading variables as scalars:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 8 \\ 1 \\ 0 \\ 11 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -5 \\ -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -7 \\ 0 \\ -3 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The first column, $u = [9, 0, 8, 1, 0, 11, 0]^T$ is one particular (i.e., fully specified) solution to $Ax = b$. What do the remaining columns produce? Notice that when we collect all of the possible solutions, by varying the values of x_2, x_5 and x_7, we are building a set that is the **span** of the three columns. The last three columns, in fact, span the set of solutions to $Ax = \mathbf{0}$. Thus every solution to $Ax = b$ is of the form $u + v$ where u is a particular solution to $Ax = b$ and v is in the set

$$\begin{aligned} & \mathit{span}\left\{[3,1,0,0,0,0,0]^T, [-4,0,-5,-6,1,0,0]^T, [-7,0,-3,5,0,0,1]\right\} \\ = \quad & (\text{solution set for } Ax = \mathbf{0}) \end{aligned}$$

Later on in this course, we will realize that not only do the last three columns listed above span the solution set for $Ax = \mathbf{0}$, but that they also have a property called independence, which makes the set of three columns a special type of spanning set called a basis. In particular, this set of three vectors in $\mathbf{R}^7$ is a basis for the solution set to $Ax = \mathbf{0}$.

Exercise: After row operations, the system $Ax = b$ of five equations in eight variables becomes:

$$\left[\begin{array}{cccccccc|c} 1 & 3 & 0 & 0 & -4 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & -8 \\ 0 & 0 & 0 & 1 & 5 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

1. Find the general solution x for $Ax = b$.
2. Find a particular solution for $Ax = b$ and a set of vectors whose span is the solution set for $Ax = \mathbf{0}$.
3. How many solutions to $Ax = b$ have $x_2 = -4, x_5 = 3$ and $x_7 = 0$?
4. How many solutions to $Ax = b$ have $x_2 = -4$ and $x_7 = 0$?