

# Power Series and their Convergence

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## Introduction

A *power series* in  $x$  is a series whose terms are all constant multiples of integer powers of  $(x - a)$  for some fixed real number  $a$ . We say that the power series is *centered at*  $x = a$ , or that it is *expanded about*  $x = a$ . Thus

$$S(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

and

$$T(x) = e - \frac{e}{2}(x-2\pi)^2 + \frac{e}{24}(x-2\pi)^4 - \frac{e}{720}(x-2\pi)^6 + \cdots + (-1)^n \frac{e}{n!}(x-2\pi)^{2n} + \cdots$$

are both power series, the first one centered at  $x = 0$ , and the second one centered at  $x = 2\pi$ . Note also that some or even many positive integer powers of  $(x - a)$  can have zero as a coefficient. In the power series  $T(x)$ , every odd power of  $(x - 2\pi)$  has coefficient 0.

For an arbitrary real number  $a$ , a generic power series expanded about  $x = a$  can be written as

$$U(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots + a_n(x - a)^n + \cdots \quad (*)$$

where  $a_0, a_1, a_2, a_3$  and  $a_n$  are all fixed real numbers.

The first natural question to ask about a power series is where does it converge. That is, for what real numbers  $x$  is the series equal to a real number? Observe that  $S(0) = 1$ , that  $T(2\pi) = e$ , and that in general,

$$\begin{aligned} U(a) &= a_0 + a_1(a - a) + a_2(a - a)^2 + a_3(a - a)^3 + \cdots + a_n(a - a)^n + \cdots \\ &= a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots + a_n(0)^n + \cdots \\ &= a_0 + 0 + 0 + 0 + \cdots + 0 + \cdots \\ &= a_0 \end{aligned}$$

Thus a power series must converge at its center. Of course, we are much more interested in convergence when we have many nonzero terms, that is, when  $x \neq a$ . In fact, the following theorem holds:

**Theorem 1 (Radius of Convergence Cases)** *Let  $U(x)$  be the power series centered at  $x = a$  given by formula (\*). Then exactly one of the following three cases holds:*

- (i)  $U(x)$  converges only for  $x = a$ ;
- (ii)  $U(x)$  converges for every real number  $x$ ;
- (iii)  $U(x)$  converges for all  $x$  with  $|x - a| < R$  and diverges for all  $x$  with  $|x - a| > R$  for some positive number  $R$ .

The real number  $R$  in the previous theorem is called the *radius of convergence* of the power series  $U(x)$ . For convenience, we will say that  $R = 0$  in case (i), and that  $R = \infty$  in case (ii) of the previous theorem. When  $0 < R$ , the inequality  $|x - a| < R$  gives an interval of convergence:  $a - R < x < a + R$ . (We will read  $a \pm R$  as  $\pm\infty$  in the case that  $R = \infty$ .) Notice that when  $0 < R < \infty$ , the previous theorem says nothing about convergence or divergence of  $U(x)$  at the values of  $x$  such that  $|x - a| = R$ . In fact, these cases are very difficult to analyze, and we will simply accept that we do not usually know what happens when  $|x - a| = R$ .

We have already studied the geometric series, which is the series  $S(x)$  as given above. We know that the geometric series converges to  $(1 - x)^{-1}$  when  $|x - 0| = |x| < 1$  and that it diverges when  $|x| > 1$ , so  $R = 1$  for the geometric series. (In fact, we know that the series also diverges when  $|x| = 1$ .)

Recall that if the series  $\sum_{n=0}^{\infty} b_n$  converges, then  $\lim_{n \rightarrow \infty} b_n = 0$ . Thus if

$$\lim_{n \rightarrow \infty} a_n(x - a)^n \neq 0$$

then  $U(x)$  does not converge at  $x = c$ . Consider the series

$$V(x) = \sum_{n=0}^{\infty} n^n(x - 5)^n$$

Then the  $n^{\text{th}}$  term of  $V(x)$  is  $n^n(x - 5)^n$ . Notice that unless  $x = 5$ ,  $|n(x - 5)|$  will eventually be greater than 2 as  $n \rightarrow \infty$ . Thus

$$\lim_{n \rightarrow \infty} |n^n(x - 5)^n| = \lim_{n \rightarrow \infty} |n(x - 5)|^n > \lim_{n \rightarrow \infty} 2^n = +\infty$$

and hence,

$$\lim_{n \rightarrow \infty} n^n(x - 5)^n \neq 0$$

for when  $x \neq 5$ . Hence  $V(x)$  diverges when  $x \neq 5$ , so we must have  $R = 0$ .

## The Ratio Test

In general, finding the radius of convergence for a series is difficult. One important tool that we can use is called the *ratio test*.

**Theorem 2 (The Ratio Test for Power Series)** Suppose that consecutive **nonzero** general terms in a power series can be written as  $c_n [(x - a)^k]^n$  and  $c_{n+1} [(x - a)^k]^{n+1}$ . Compute the limit

$$L = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

(Notice that the coefficient of the  $n^{\text{th}}$  term is on TOP, and the coefficient of the  $(n+1)^{\text{st}}$  term is on the BOTTOM.)

If  $L$  does not exist, the test fails to tell us anything useful.

If  $L = 0$ , then the radius of convergence is  $R = 0$ .

If  $L = \infty$ , then the radius of convergence is  $R = \infty$ .

If  $0 < L < \infty$ , then the power series converges for all  $x$  satisfying  $|(x-a)^k| < L$  and diverges for all  $x$  satisfying  $|(x-a)^k| > L$ . Thus  $R = L^{1/k}$ .

Let us consider some examples.

**Example 3** For the geometric series  $S(x)$ , consecutive nonzero general terms are  $1x^n$  and  $1x^{n+1}$ . Hence  $L = \lim_{n \rightarrow \infty} \left| \frac{1}{1} \right| = 1$ . Thus  $R = 1$ , as we already know.

**Example 4** For the series  $V(x)$  given above, consecutive nonzero general terms are  $n^n(x-5)^n$  and  $(n+1)^{n+1}(x-5)^{n+1}$ . Thus

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n (n+1)} = \lim_{n \rightarrow \infty} \left[ \frac{n^n}{n^n \left(1 + \frac{1}{n}\right)^n} \cdot \frac{1}{n+1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \frac{1}{n+1} \right] = \left[ \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right] \left[ \lim_{n \rightarrow \infty} \frac{1}{n+1} \right] = \frac{1}{e} \cdot 0 = 0 \end{aligned}$$

Thus  $R = 0$  for  $V(x)$  as we saw earlier.

**Example 5** For the power series

$$W(x) = \frac{2}{3}(x-5)^3 - \frac{4}{4}(x-5)^6 + \frac{8}{5}(x-5)^9 + \cdots + (-1)^n \frac{2^n}{n+2}(x-5)^{3n} + \cdots$$

consecutive nonzero general terms are

$$(-1)^{n+1} \frac{2^n}{n+2}(x-5)^{3n} = (-1)^{n+1} \frac{2^n}{n+2} [(x-5)^3]^n$$

and

$$(-1)^{(n+1)+1} \frac{2^{n+1}}{(n+1)+2}(x-5)^{3(n+1)} = (-1)^{(n+1)+1} \frac{2^{n+1}}{(n+1)+2} [(x-5)^3]^{n+1}$$

Thus

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{2^n}{n+2}}{(-1)^{(n+1)+1} \frac{2^{n+1}}{(n+1)+2}} \right| = \lim_{n \rightarrow \infty} \frac{\frac{2^n}{n+2}}{\frac{2^{n+1}}{(n+1)+2}} \\ &= \lim_{n \rightarrow \infty} \frac{2^n [(n+1)+2]}{2^{n+1}(n+2)} = \lim_{n \rightarrow \infty} \frac{(n+3)}{2(n+2)} = \lim_{n \rightarrow \infty} \frac{n(1 + \frac{3}{n})}{2n(1 + \frac{2}{n})} = \frac{1}{2} \end{aligned}$$

Thus  $W(x)$  converges for  $|(x-5)^3| < \frac{1}{2}$ , which is to say, for  $|x-5| < (1/2)^{1/3}$ . Then  $W(x)$  converges for all  $x$  in the open interval  $5 \pm (1/2)^{1/3}$ , and it diverges when either  $x < 5 - (1/2)^{1/3}$  or else  $x > 5 + (1/2)^{1/3}$ . (We do not know what happens when  $x = 5 \pm (1/2)^{1/3}$ .)

**Example 6** For the power series

$$Y(x) = 1 - \frac{2}{1}(x-5)^3 - \frac{4}{2}(x-5)^6 + \frac{8}{6}(x-5)^9 + \cdots + (-1)^n \frac{2^n}{n!}(x-5)^{3n} + \cdots$$

consecutive nonzero general terms are

$$(-1)^n \frac{2^n}{n!}(x-5)^{3n} = (-1)^n \frac{2^n}{n!} [(x-5)^3]^n$$

and

$$(-1)^{n+1} \frac{2^{n+1}}{(n+1)!}(x-5)^{3(n+1)} = (-1)^{n+1} \frac{2^{n+1}}{(n+1)!} [(x-5)^3]^{n+1}$$

Thus

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n \frac{2^n}{n!}}{(-1)^{n+1} \frac{2^{n+1}}{(n+1)!}} \right| = \lim_{n \rightarrow \infty} \frac{\frac{2^n}{n!}}{\frac{2^{n+1}}{(n+1)!}} \\ &= \lim_{n \rightarrow \infty} \frac{2^n(n+1)!}{2^{n+1}n!} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot n!}{2 \cdot n!} = \lim_{n \rightarrow \infty} \frac{n+1}{2} = \infty \end{aligned}$$

Hence, the radius of convergence for  $Y(x)$  is  $R = \infty$ . That is,  $Y(x)$  converges for every real  $x$ .

Although the ratio test tells us for what values of  $x$  the power series converges, it does NOT tell us to what function the power series converges. Thus we do not know a formula for either  $W(x)$  or  $Y(x)$  even though we know where they converge.

## New Power Series from Old Power Series

In the notes on the geometric series, which is an example of a power series, we saw that we could obtain new series by algebraic manipulations, by substitutions, by integration and by differentiation. We can apply these techniques to all power series.

For algebraic manipulations and substitutions, the key thing to remember is that if we have a power series  $U(x)$  centered at  $x = a$  with radius of convergence  $R$ , and if we replace  $x$  by some expression  $g(y)$ , then we need to recompute our center and our radius by starting with  $|x - a| < R$ , and changing it to  $|g(y) - a| < R$ . For example, if we have  $a = 10$  and  $R = 30$ , so that  $|x - 10| < 30$

is the interval of convergence for  $U(x)$ , and we replace  $x$  with  $5(y-7)$ , then the new series,  $U(5(y-7))$  converges when  $|5(y-7)-10| < 30$ , which is to say, when  $|y-(7+\frac{10}{5})| < \frac{30}{5}$ . Thus the new series has center  $7+\frac{10}{5} = 9$ , and radius of convergence  $\frac{30}{5} = 6$ .

For differentiation and integration, we can extend the sum rules for integration and differentiation of a finite number of summands to rules for infinitely many summands.

**Theorem 7 (Differentiation and Integration of Power Series)** *Suppose that  $U(x)$  is a power series centered about  $x = a$  as given by formula (\*) with radius of convergence  $R > 0$ . Then*

$$U'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots + na_n(x-a)^{n-1} + \cdots$$

and

$$U''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a)^2 + 3 \cdot 4a_4(x-a)^2 \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots$$

and so on for higher order derivatives. Further,

$$\int U(x) dx = K + a_0(x-a) + a_1 \frac{(x-a)^2}{2} + a_2 \frac{(x-a)^3}{3} + \cdots + a_n \frac{(x-a)^{n+1}}{n+1} + \cdots$$

for some constant  $K$ , and

$$\int_a^x U(t) dt = a_0(x-a) + a_1 \frac{(x-a)^2}{2} + a_2 \frac{(x-a)^3}{3} + \cdots + a_n \frac{(x-a)^{n+1}}{n+1} + \cdots$$

(Note that the lower limit of the integral is the value of the center,  $x = a$ .) Finally, the series for the integrals and the series for the derivatives all converge for  $|x-a| < R$ , and in the case that  $R < \infty$ , they all diverge for  $|x-a| > R$ . For  $0 < R < \infty$ , the convergence or divergence of each of the series must be tested separately for each  $x$  value for which  $|x-a| = R$ .

**Example 8** Consider the power series  $Y(x)$  given above. Since  $Y(x)$  converges for all  $x$ , so will all integrals and derivatives of  $Y(x)$ . Observe that

$$Y'(x) = 0 - \frac{2}{1} \cdot 3(x-5)^2 - \frac{4}{2} \cdot 6(x-5)^5 + \frac{8}{6} \cdot 9(x-5)^8 + \cdots + (-1)^n \frac{2^n}{n!} \cdot 3n(x-5)^{3n-1} + \cdots$$

and

$$\int_5^x Y(t) dt = 1(x-5) - \frac{2}{1} \frac{(x-5)^4}{4} - \frac{4}{2} \frac{(x-5)^7}{7} + \frac{8}{6} \frac{(x-5)^{10}}{10} + \cdots + (-1)^n \frac{2^n}{n!} \frac{(x-5)^{3n+1}}{3n+1} + \cdots$$

**Example 9** The function  $e^x$  has a power series centered at  $x = 0$  with radius of convergence  $R = \infty$ . Specifically,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$$

We can use this to get a power series for

$$F(x) = \int_0^x \exp\left(-\frac{t^2}{2}\right) dt$$

First, by substituting  $x = -t^2/2$  into the power series for  $e^x$ , we get

$$\begin{aligned} \exp\left(-\frac{t^2}{2}\right) &= 1 + \left(-\frac{t^2}{2}\right) + \frac{1}{2} \left(-\frac{t^2}{2}\right)^2 + \frac{1}{6} \left(-\frac{t^2}{2}\right)^3 + \frac{1}{24} \left(-\frac{t^2}{2}\right)^4 + \cdots + \frac{1}{n!} \left(-\frac{t^2}{2}\right)^n + \cdots \\ &= 1 - \frac{t^2}{2} + \frac{1}{2} \cdot \frac{t^4}{2^2} - \frac{1}{6} \cdot \frac{t^6}{2^3} + \frac{1}{24} \cdot \frac{t^8}{2^4} + \cdots + \frac{(-1)^n}{2^n n!} t^{2n} + \cdots \end{aligned}$$

so

$$\begin{aligned} F(x) &= \int_0^x \exp\left(-\frac{t^2}{2}\right) dt \\ &= \int_0^x \left(1 - \frac{t^2}{2} + \frac{1}{2} \cdot \frac{t^4}{2^2} - \frac{1}{6} \cdot \frac{t^6}{2^3} + \frac{1}{24} \cdot \frac{t^8}{2^4} + \cdots + \frac{(-1)^n}{2^n n!} t^{2n} + \cdots\right) dt \\ &= x - \frac{x^3}{2 \cdot 3} + \frac{1}{2} \cdot \frac{x^5}{2^2 \cdot 5} - \frac{1}{6} \cdot \frac{x^7}{2^3 \cdot 7} + \frac{1}{24} \cdot \frac{x^9}{2^4 \cdot 9} + \cdots + \frac{(-1)^n}{2^n n!} \cdot \frac{x^{2n+1}}{2n+1} + \cdots \\ &= x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{1}{336}x^7 + \frac{1}{3456}x^9 + \cdots + \frac{(-1)^n}{2^n n!} \cdot \frac{x^{2n+1}}{2n+1} + \cdots \end{aligned}$$

## Exercises

- Using the power series for  $e^x$  given in Example 9, confirm that the radius of convergence is  $R = \infty$ .
- Differentiate the power series for  $e^x$  given in Example 9. Is the derivative of the series also the series for  $e^x$ ?
- Integrate the power series for  $e^x$  given in Example 9 from 0 to  $t$ . Do you get power series for the the function that you expect from  $\int_0^t e^x dx$ ?
- Use the power series for  $e^x$  given in Example 9 to find a power series centered at  $x = 0$  for  $e^{-x}$ .
- Use substitution to find a power series for  $e^{x-7}$  centered at  $x = 7$ . Find the radius of convergence for this new power series.
- Use the power series for  $e^{x-7}$  that you found in the previous exercise to find a power series for  $e^x$  that is centered at  $x = 7$ . (Hint: This has a one line answer.)

7. The probability that a standard (mean 0, variance 1) normal random variable  $Z$  lies between the numbers  $a$  and  $b$  is the given by  $\Pr(a \leq Z \leq b) = \Phi(b) - \Phi(a)$  where

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{t^2}{2}\right) dt.$$

In Example 9 we found a power series for  $\sqrt{2\pi} \cdot [\Phi(x) - \frac{1}{2}]$ , and we explicitly found the first five terms of the power series.

- Find a power series centered at  $x = 0$  for  $\Phi(x)$ .
  - Using only the first three terms of the power series for  $\Phi(x)$ , estimate  $\Phi(0.1)$ ,  $\Phi(0.5)$ ,  $\Phi(1)$  and  $\Phi(2)$ .
  - Using only the first five terms of the power series for  $\Phi(x)$ , estimate  $\Phi(0.1)$ ,  $\Phi(0.5)$ ,  $\Phi(1)$  and  $\Phi(2)$ .
  - Using software or a normal probability table together with the result from part (a), obtain values for  $\Phi(0.1)$ ,  $\Phi(0.5)$ ,  $\Phi(1)$  and  $\Phi(2)$ .
  - Compare the results from parts (b), (c) and (d), and briefly comment on what you observe.
  - Show that the next nonzero term in the power series for  $F(x)$  is  $-(42240)^{-1}x^{11}$ , and use that to find the  $x^{11}$  term in the power series for  $\Phi(x)$ . Does including the additional term in your estimates of  $\Phi(0.1)$ ,  $\Phi(0.5)$ ,  $\Phi(1)$  and  $\Phi(2)$  change any of your estimates? Which ones? Does it improve those estimates?
8. Examine the power series

$$A(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- Write out at least the first three nonzero terms of each series.
  - Find the radius of convergence for each series.
  - Compute the power series centered at  $x = 0$  for  $A'(x)$  and  $B'(x)$ , and write out at least the first three nonzero terms of each.
  - Compute and carefully simplify  $A'(x) + B(x)$ .
  - Compute and carefully simplify  $A(x) - B'(x)$ .
  - Compute and carefully simplify  $A''(x) + A(x)$ .
9. BONUS: Using the fact that  $i^2 = -1$ ,  $i^3 = -i$ , and  $i^4 = 1$ , compute and simplify the power series centered at  $x = 0$  for  $e^{ix} - A(x) - iB(x)$ , where  $A(x)$  and  $B(x)$  are the series given in the previous exercise. (Hint: It might help to write out the first eight nonzero terms in the series for  $e^{ix}$  and the first four nonzero terms in each of the series for  $A(x)$  and  $B(x)$ .)

10. BONUS: If you know about the arithmetic of complex numbers, then you know that the absolute value of a real number is a special case of the magnitude of a complex number. Suppose that  $z$  is an unknown complex number, that  $a$  is a fixed complex number, and that  $R$  is a positive real number. Describe the set  $|z - a| = R$  in the complex plane. For example, what geometric figure is described by the points satisfying  $|z - 0| = 1$ ? Relate this to why  $R$  might be called the *radius* of convergence.