# Teach ill-conditioning to introductory linear algebra students in a single lecture!

## Shake a Stick at Ill-Conditioning – A Multi-Modal Lesson

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# A Common Student Misconception

Held by students

- Post College Algebra
- Post Calculus
- Post Linear Algebra
- Post BA with Math Major

The Misconception –

**All** invertible linear systems of equations behave the same way:

- There is a unique solution
- Gaussian Elimination gives it
- Computers always do GE faster and better
- there is no more to be said

# Equipment for Physical Demonstration

Required Equipment:

- Chalk board and chalk (white board and markers)
- Two chairs
- Two rigid rods at least six feet long
  - Broom sticks or mop handles
  - Yard or meter sticks duct-taped together
  - Rigid plastic pipe
  - The longer, the better!!
- Five student "volunteers" (Two with decent balance)

# Before The Physical Demonstration

- Discuss idea of data uncertainty when using real world information
- Discuss geometric implication in linear context
  - uncertainty of line location
  - an envelope of possible lines
- Sketch a family of lines and the envelope of a "Shaking Line"
- Discuss two uncertain lines and their intersection
  - A region of possible intersection points
- Discuss geometric effect of right hand side perturbations on systems and the impact on the intersection point

# Perform The Physical Demonstration - Shake the Sticks!

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  - *First time* with rods crossing at approximately a right angle but NOT horizontal and vertical
  - *Second time* with rods crossing but almost parallel.
- Each end of a stick is held by a different student.

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- Perform at least a dozen steps of shaking and marking, more as needed in order for patterns to appear for case of nearly parallel sticks.
- Time permitting, repeat demonstration using different orientation for the lines or different angle between the lines.

# The Computational Experiment - Part 1

- Linear system $AX = B$ with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

- Solve it and sketch the lines.
- Note lines are perpendicular.

- New plot (#1) – plot the right-hand side vector $B$ and eight neighboring points $\widehat{B}$ given by

$$\begin{bmatrix} 1 \pm 0.01 \\ 3 \end{bmatrix}, \begin{bmatrix} 0.99 \\ 3 \pm 0.01 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \pm 0.01 \end{bmatrix}, \begin{bmatrix} 1.01 \\ 3 \pm 0.01 \end{bmatrix}$$

- Note all entries in these points agree within 1% with entries of $B$.
- Have each student solve at least one of new problems $AX = \widehat{B}$.
- Previously prepared handouts each with two of the systems help.
- Plot student solutions along with original on new plot (#2).
- Note solutions to perturbed problems are all quite close to original solution.
- Share plot of 3000 small, random perturbations of $B$ and plot of corresponding solutions (#3 and #4).

Figure 1. The point B = [1,3]' and its eight neighboring points.

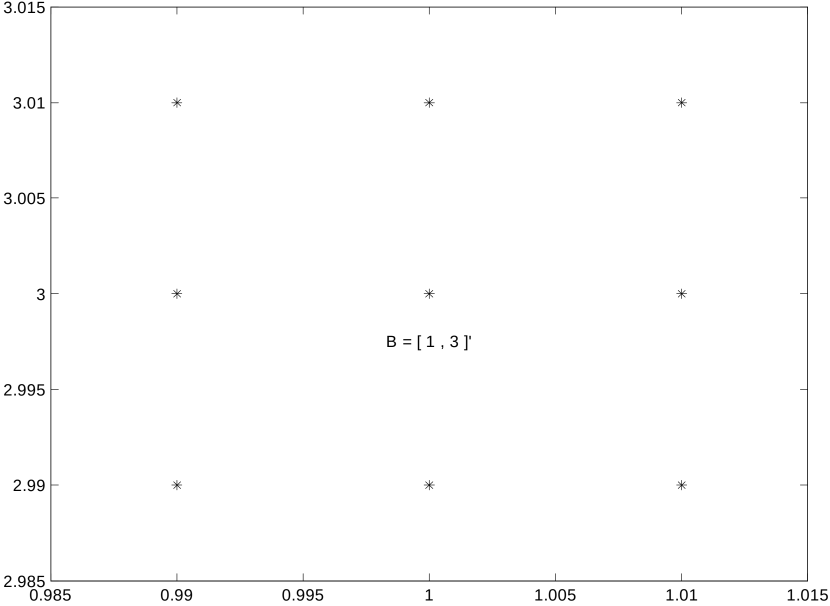


Figure 2. The solutions to the linear system for B and its eight neighbors.

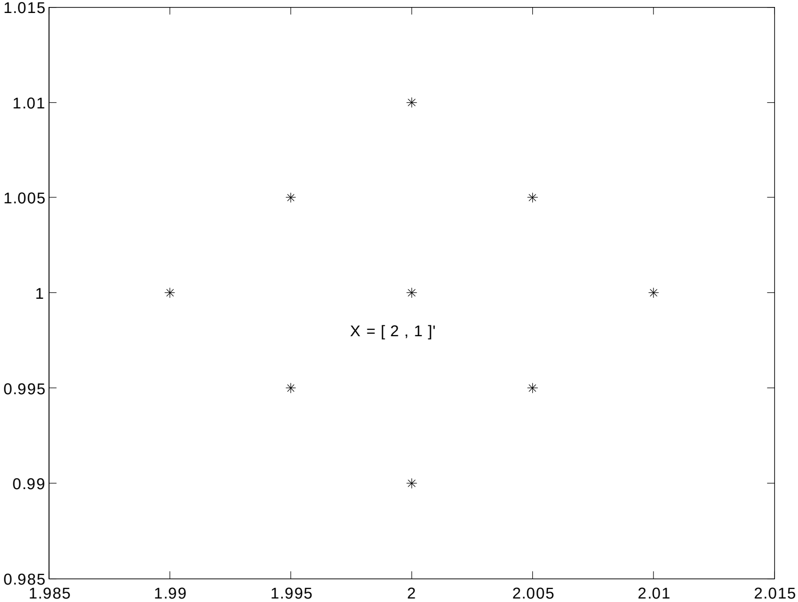


Figure 3. Three thousand perturbations of B = [1,2]'.

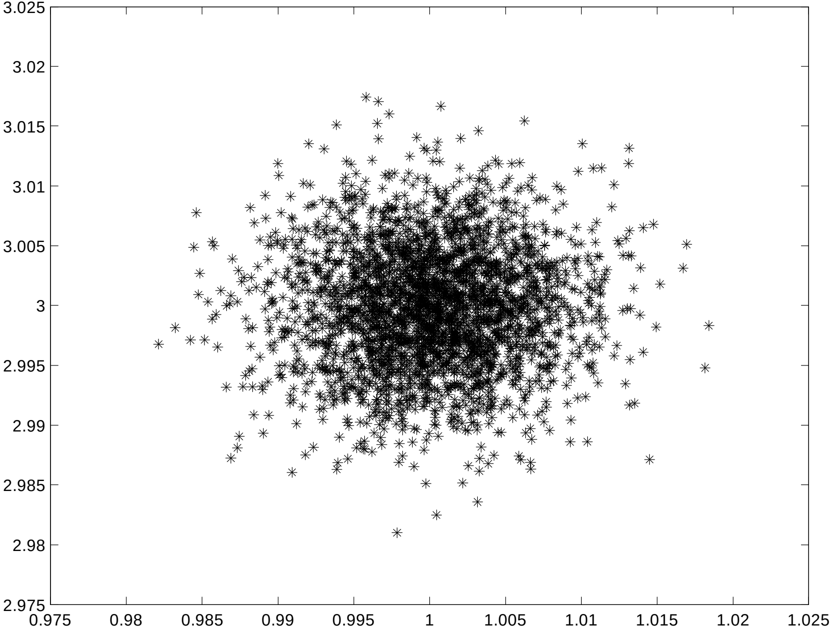
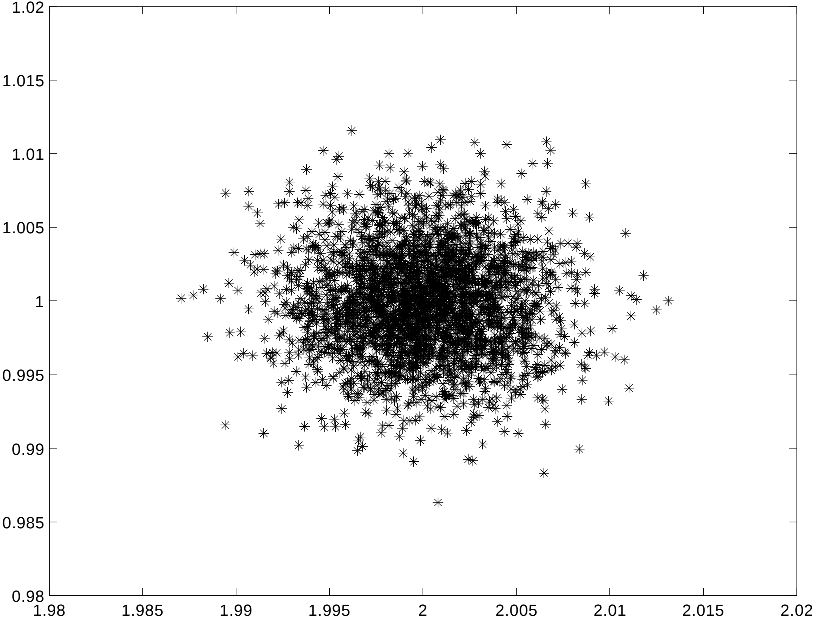


Figure 4. The solutions for the three thousand perturbed systems.



# The Computational Experiment - Part 2

- Linear system $AX = B$ with

$$A = \begin{bmatrix} 1 & 1.0001 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3.0001 \\ 3 \end{bmatrix}.$$

- Solve system and sketch the lines.
- Note lines are almost parallel.
- Repeat process of perturbations to $B$ using same $\Delta b_j = \pm 0.01$.
- Have students solve perturbed systems.
- Plot perturbed solutions (#5 and #6).
- Handouts simplify organizational details in a larger class.

Figure 5. The point B = [3.0001,3]' and its eight neighbors.

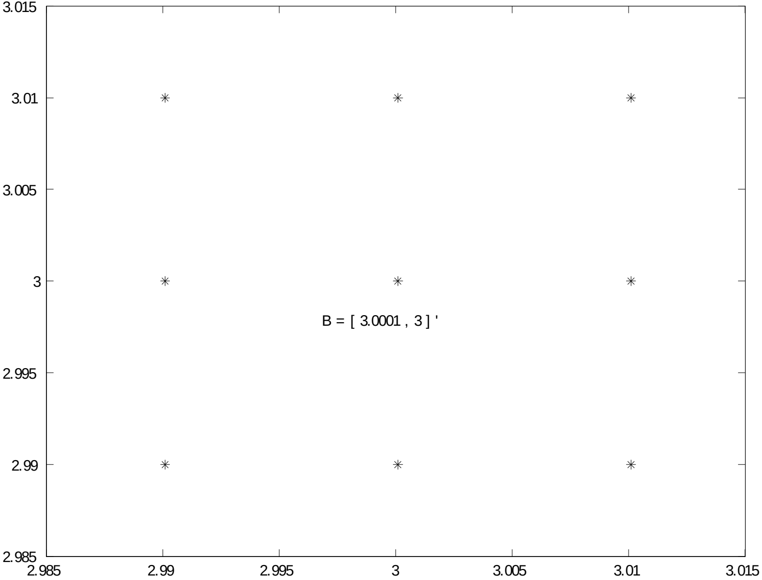
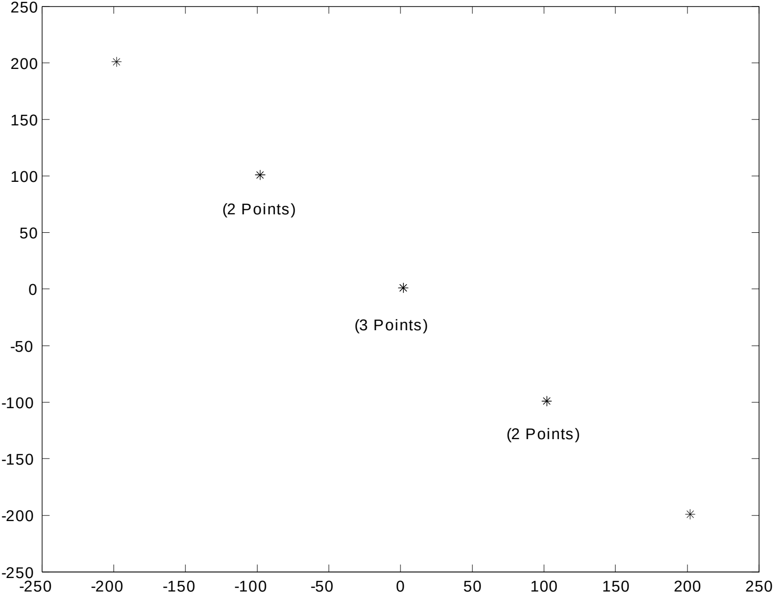


Figure 6. The solutions to the linear system for B and its eight neighbors.



- Comment on the very different outcome.
- Note some right-hand side perturbations produce small perturbations in solutions but others produce large ones.
- Share plot of 3000 random perturbations of $B$ and plot of corresponding solutions (#7 & #8).

Figure 7. Three thousand perturbations of B = [3.0001,3]'.

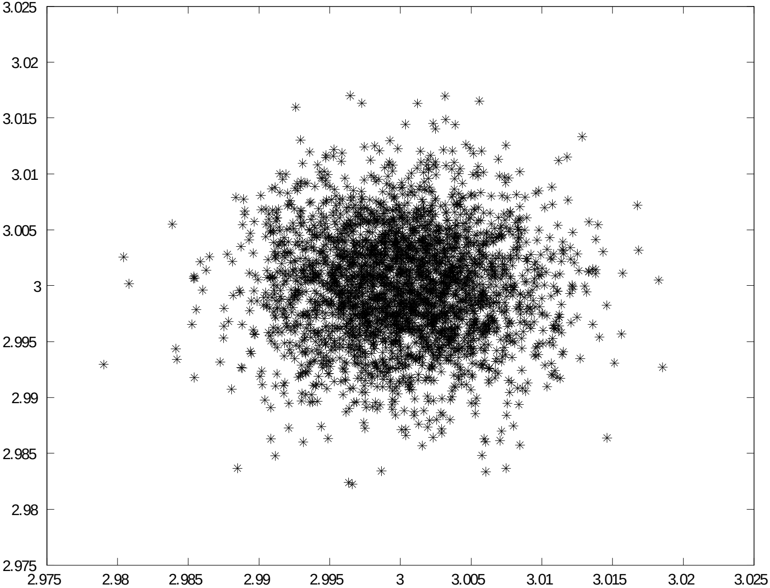
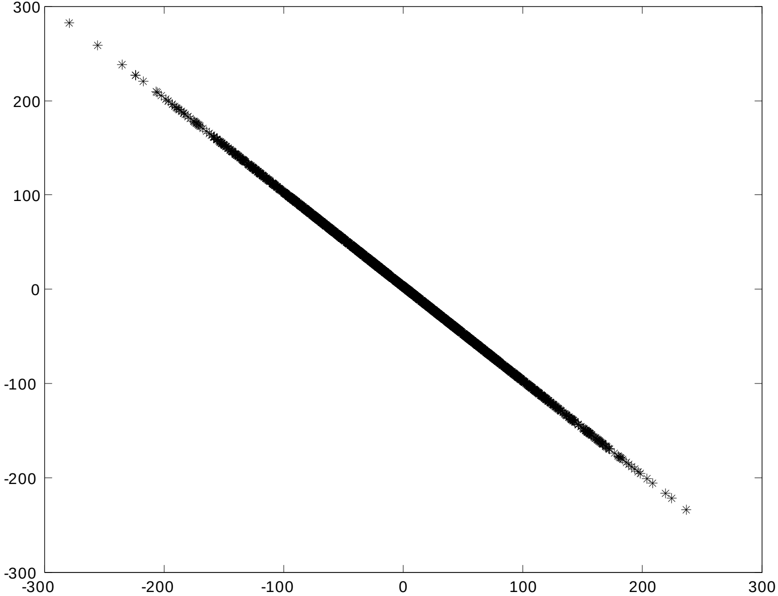


Figure 8. The solutions for the three thousand perturbed systems.



# The Algebraic Analysis - Part 1

- Solve the $1^{st}$ system $AX = B + \Delta B$ algebraically.
- $$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + \Delta b_1 \\ 3 + \Delta b_2 \end{bmatrix},$$

  Solution

  $$x = 2 + \frac{\Delta b_1 + \Delta b_1}{2},$$
  $$y = 1 + \frac{\Delta b_2 - \Delta b_1}{2}.$$

- Note that **small changes** in $\Delta b_1$ and $\Delta b_2$ on right-hand side lead to **small changes** in $x$ and $y$.

# The Algebraic Analysis - Part 2

- Solve the $2^{nd}$ system $AX = B + \Delta B$ algebraically.
- $$\begin{bmatrix} 1 & 1.0001 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.0001 + \Delta b_1 \\ 3 + \Delta b_2 \end{bmatrix},$$
- Solution
$$\begin{aligned} x &= 2 + \Delta b_2 - 10000(\Delta b_1 - \Delta b_2), \\ y &= 1 + 10000(\Delta b_1 - \Delta b_2). \end{aligned}$$

- When $\Delta b_1 = \Delta b_2$, **small changes** in $\Delta b_1$ lead to **small changes** in $x$ and **no changes** in $y$.
- When $\Delta b_1 \neq \Delta b_2$, **small changes** in $\Delta b_1$ and $\Delta b_2$ can produce **very large changes** in $x$ and $y$.
- Emphasize **only some small right hand side perturbations** lead to **large solution perturbations**.

# Concluding Remarks

- Emphasize that as linear system changes from roughly orthogonal to roughly degenerate, system becomes more sensitive to perturbations.
- Prompt students for intuitive discussion of what happens with three linear systems in three unknowns, being sure to discuss geometric meaning of perturbing equation of a plane.
- Emphasize quality of the solutions of linear system based on uncertain data depends not just on quality of data, but also on geometry of the system.
- Reemphasize even with good data, the angles between the lines or planes can affect certainty that one should assign to the solution.

# Further Adventures

- *condition number* as a numerical measure of geometric effects and as a measure of worst-case error magnification.
  - *well-conditioned* ("small" condition number)
  - *ill-conditioned* ("big" condition number)
  - For $1^{st}$ system $AX = B$, condition number of $A$ is 1. (Well-Conditioned)
  - For $2^{nd}$ system $AX = B$, condition number of $A$ is 40002. (Ill-Conditioned)
- Even perpendicular lines can be ill-conditioned by scaling.

- Have students investigate well scaled and badly scaled systems such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + \Delta b_1 \\ 3 + \Delta b_2 \end{bmatrix}$$

  and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.0001 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + \Delta b_1 \\ 0.0003 + \Delta b_2 \end{bmatrix}.$$

- Have students conduct numerical experiments that involve perturbations of both $A$ and $B$.
- Using computers as well as geometric insight, have students investigate perturbation sensitivity in the case of three equations in three unknowns.

Thank you!

The plots and the paper will be available on the author's website:

http://www.plu.edu/~stuartjl/PLUCourses/shake_a_stick/

Just Google: "Stuart" AND "Shake a Stick"