The 10 Fundamental Properties of Real n-Space

The symbols $\mathbb{R}^n$ denote the set of all ordered n-tuples of real numbers. This set with two entrywise operations, scalar multiplication and vector addition, is sometimes called *real n-space*, or simply n-space. This set and its operations satisfy 10 fundamental properties:

1. For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$.
   (The set $\mathbb{R}^n$ is closed under vector addition.)
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
   (Vector addition is associative.)
3. For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
   (Vector addition is commutative.)
4. There is a zero vector $\mathbf{0}$ in $\mathbb{R}^n$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.
   ($\mathbb{R}^n$ contains an additive identity element.)
5. For each $\mathbf{u} \in \mathbb{R}^n$, there is $-\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
   (Each vector in $\mathbb{R}^n$ has an additive inverse in $\mathbb{R}^n$.)
6. For all $c \in \mathbb{R}$ and all $\mathbf{u} \in \mathbb{R}^n$, $c\mathbf{u} \in \mathbb{R}^n$.
   (The set $\mathbb{R}^n$ is closed under scalar multiplication.)
7. For each $\mathbf{u} \in \mathbb{R}^n$, $1\mathbf{u} = \mathbf{u}$.
8. For all $c \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
9. For all $c, d \in \mathbb{R}$ and all $\mathbf{u} \in \mathbb{R}^n$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
10. For all $c, d \in \mathbb{R}$ and all $\mathbf{u} \in \mathbb{R}^n$, $c(d\mathbf{u}) = (cd)\mathbf{u}$.

Properties 8 and 9 are called *distributive laws.*

We have proven in class that there are two additional properties of $\mathbb{R}^n$ that are direct consequences of these ten properties:

11. For all $\mathbf{u} \in \mathbb{R}^n$, $(-1)\mathbf{u} = -\mathbf{u}$.
12. For all $\mathbf{u} \in \mathbb{R}^n$, $0\mathbf{u} = \mathbf{0}$.

We have proven in class that any subset of $\mathbb{R}^n$ automatically satisfies Properties $2, 3, 7-12$. Thus if we want to show that we can replace $\mathbb{R}^n$ in every property statement with S where S is a nonempty subset of $\mathbb{R}^n$, we only have to check that properties $1, 4, 5, 6$ hold. Properties 6 and 11 imply Property 5. Properties 6 and 12 imply that Property 4 holds **when S is nonempty**. Thus, to show that a subset S of $\mathbb{R}^n$ satisfies all ten fundamental properties, we only need to check that S is nonempty and that properties 1 and 6 hold for S. That is, we need to only check the following three properties: (i) S is nonempty, (ii) S is closed under vector addition, and (iii) S is closed under scalar multiplication.

Any subset S of $\mathbb{R}^n$ that satisfies these last three properties is called a *subspace* of $\mathbb{R}^n$, and must actually satisfy all twelve listed properties above. Notice that if $\mathbf{0} \in S$, then clearly S is nonempty. Conversely, if conditions $(i) - (iii)$ hold, then Property 4 must hold, so $\mathbf{0} \in S$. Consequently when properties (ii) and (iii) hold, property (i) "S is nonempty" is equivalent to the property "$\mathbf{0} \in S$". This is why some authors define S to be a subspace of $\mathbb{R}^n$ if it satisfies the three properties: (i) $\mathbf{0} \in S$, (ii) S is closed under vector addition, and (iii) S is closed under scalar multiplication. As a general principle, when testing a set S to see whether or not it is a subspace, begin by checking that $\mathbf{0} \in S$.