

Universal Statements and Counterexamples

A universal statement is a mathematical statement that is supposed to be true about all members of a set. That is, it is a statement such as, "For all $x \geq 7$, $\frac{1}{x} < \frac{1}{2}$," or "The square of a real number is nonnegative." . As these two examples show, universal statements can explicitly contain universal quantifiers ("all"), or the universal quantifiers can be implicit. Universal quantifiers are words such as "all", "every" and "each". The following statements about all numbers in the set of real numbers convey the same information:

- The square of a real number is nonnegative.
- The square of each real number is nonnegative.
- The square of every real number is nonnegative.
- The squares of all real numbers are nonnegative.
- For every (each, all) $a \in \mathbb{R}$, $a^2 \geq 0$.

Notice in this context that the first statement has the potential to be confusing. Did the author mean that every real number has a nonnegative square, or did the author mean that there is *some* real number with a nonnegative square? Because of this potential for confusion, you should be careful to include universal quantifier words when writing universal statements. Also because of this potential for confusion, it is important that you carefully read mathematical statements to see if there are implicit universal quantifiers.

Some authors use "any" as a universal quantifier, writing "The square of any real number is nonnegative." Unfortunately, this conflicts with a different use of "any" as in "Are there any solutions to $\cos(x) = 1$?", the answer to which is, "Yes, $x = 0$ is such a solution." As a consequence of the ambiguities associated with "any", you should not use "any" as a universal quantifier.

Consider a few more examples of true universal statements that convey the same information:

- For all $c \in \mathbb{R}$ and all $\mathbf{u} \in \mathbb{R}^n$, $c\mathbf{u} \in \mathbb{R}^n$.
- For each $c \in \mathbb{R}$ and each $\mathbf{u} \in \mathbb{R}^n$, $c\mathbf{u} \in \mathbb{R}^n$.
- For every $c \in \mathbb{R}$ and every $\mathbf{u} \in \mathbb{R}^n$, $c\mathbf{u} \in \mathbb{R}^n$.
- Scalar multiplication of a vector in \mathbb{R}^n always produces a vector in \mathbb{R}^n .

Why is the last statement universal? Because "always" contains the universal quantifier "all".

Compare the two statements, "The square of a real number is nonnegative," and "The square of a real number equals the number itself." The first is a universal statement with an implied "every". What about the second statement? It actually is a restricted statement, and should be quantified as follows:

"The square of *some* real number equals the number itself." In this second formulation, it is clearer that the author does **not** mean that every real number equals its square. Notice that the restricted statement is clarified by using the restricted quantifier "some".

How do we assess the truth of a universal statement? **Either we show that it is always true (a proof), or we show that it is false in at least one instance. Such an instance is a counterexample.** A counterexample should be as explicit and particular as possible. Consider the statement: "If a function has a zero derivative at $x = a$, then it must have a local minimum or maximum at $x = a$." This is actually a universal statement since it can be rewritten as: "Every function that has a zero derivative at $x = a$ has a local minimum or maximum at $x = a$." We might try to prove that this is true for all functions that have zero derivatives at $x = a$, or we might try to show that the statement is false. In fact, the statement is false. To establish its falsity, we only have to find one example of a function that has a zero derivative at $x = a$, but which does not have a maximum or minimum at $x = a$. Further, we want to be as specific about such a function as possible so that it is immediately obvious that we have a function that disobeys the claimed fact. A good counterexample here would be $f(x) = (x - a)^3$ since $f'(a) = 0$ but $x = a$ is neither a local minimum nor a local maximum. A less good counterexample would be $f(x) = (x - a)^n$ where n is an odd integer with $n \geq 3$. A poor counterexample would be $f(x) = -a^3b^2x^5 - 2a^3bx^6 - a^3x^7 + 3a^2b^2x^6 + 6a^2bx^7 + 3a^2x^8 - 3ab^2x^7 - 6abx^8 - 3ax^9 + b^2x^8 + 2bx^9 + x^{10}$ where b is some fixed real number.

Consider the universal statement, "For all $x > 7$, $x^2 \leq 100$." It does not matter that we can find lots of choices of x that satisfy both $x > 7$ and $x^2 \leq 100$. What matters is that we can find at least one x for which $x > 7$ and $x^2 > 100$. For the given statement, $x = 11$ is a great counterexample, and $x = 1,000,000$ is a good counterexample. While $x = \pi^e$ is a valid counterexample, it is a poor counterexample because it is not obvious that $\pi^e > 7$ and $(\pi^e)^2 > 100$.

In summary,

- A well-written universal statement should include one of the universal quantifiers: all, every, each.
- A false universal statement is shown to be false by providing a good counterexample.